Forests & Groves: Minimal Gauge Invariant Classes of Tree Diagrams in Gauge Theories

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Motivation

- **theoretical curiosity:**
  - How do gauge theories do their "magic"?
  - cancellations of unphysical contributions remain amazing in practical calculations
  - ... despite the abstract arguments ...
- **practical importance:**
  - phenomenology needs well behaved matrix elements & cross sections, if possible compact
  - systematic procedures desirable
  - ... Monte Carlo event generator generators ...
- **short term goal:**
  - automagic selection of gauge invariant subsets of Feynman diagrams
- **long term goal:**
  - automagic calculation of compact expressions with builtin gauge cancellations

Phenomenology: weak interactions exchange vector (& pseudo-vector) quantum numbers:

∴ unphysical degrees of freedom are present (time-like "polarization" states of intermediate vector bosons)

∴ gauge theories are the only known consistent description of interacting vector particles, in which the unphysical degrees of freedom cancel

Perturbative calculations require **gauge fixing:**

∴ intricate cancellations: either of

- bad high energy behaviour in unitarity gauge

or of

- unitarity violating contributions in R_ξ-gauges

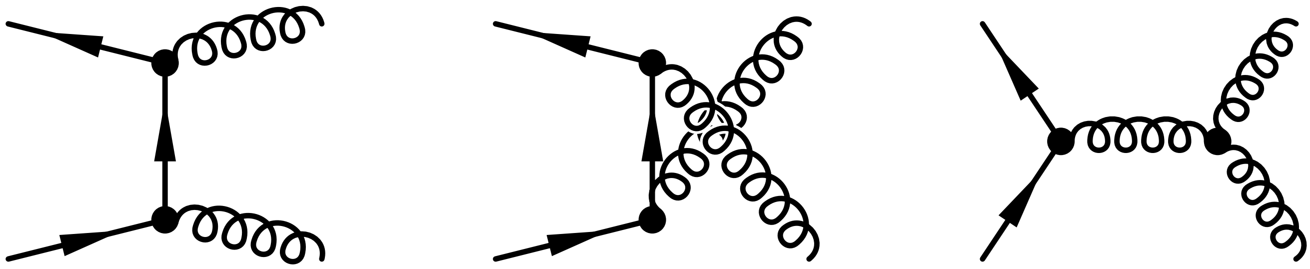
Renormalizability & unitarity best proved abstractly

1. prove quantum action principles (power counting rules) of BPHZ subtracted theory: gauge invariance broken & unitarity violated
2. prove that there is no obstruction to restoring BRS invariance with finite counterterms of dimension-4, other than the ABBJ-anomaly
3. prove unitarity with quartet-mechanism

∵ BPHZ/QAP/BRS procedure decouples perturbative power counting & gauge structure

∴ implementation in concrete calculations is **not** obvious

∵ Ward identities relate diagrams with different pole structure



Reason: numerator factors can cancel parts of denominators

∴ how to disentangle the intricate web of kinematical structure & gauge structure?

∴ do we always have to calculate **all** diagrams to avoid being "a little bit gauge invariant"?

Practical consideration: it is more economical to spent the time improving Monte Carlo statistics for the important pieces of the amplitude than to calculate the complete amplitude all the time!

The interesting flavor physics at the LHC & the Linear Collider is different from QCD with light quarks:

- the gauge symmetry is broken spontaneously

$\because$ the gauge group is not simple & the factors are mixed

$$\begin{pmatrix} Z^0 \\ A \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} W^3 \\ B \end{pmatrix}$$

$\therefore$ combinatorics more intricate

$\because$ the flavor of leptons & heavy quarks is tagged & the flavor transformations are **not** orthogonal on the gauge transformations

$\therefore$ no summation over complete gauge multiplets possible

$\therefore$ many amplitudes not gauge **invariant**, only gauge **covariant**

☹ QCD recursion relations or string inspired methods not directly applicable

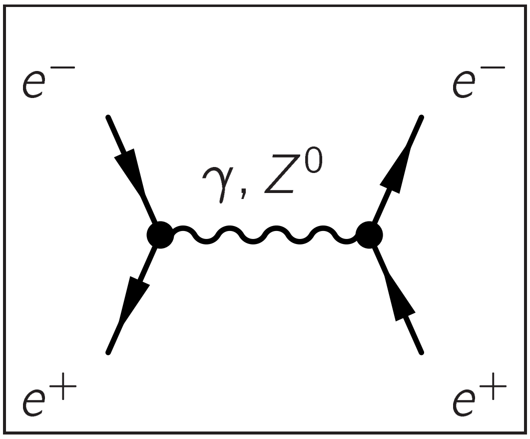
- use physical final states

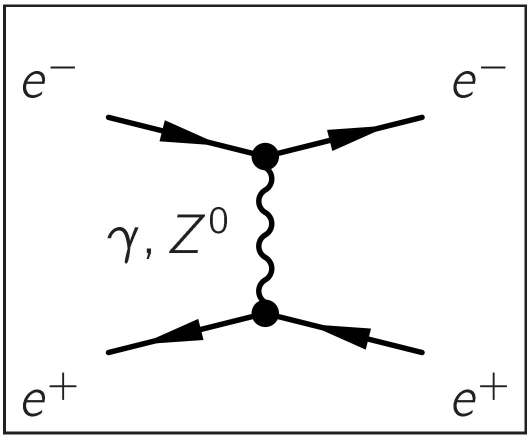
  ∵ there are 10 tree diagrams contributing to $e^+e^- \to \mu^-\bar\nu_\mu u\bar d$

  ∴ the corresponding subset of the 20 diagrams for $e^+e^- \to e^-\bar\nu_e u\bar d$ must be gauge invariant

∴ flavor selection rules appear to be a good criterion

- simplest example: s-channel & t-channel separately gauge-invariant in Bhabha-scattering:

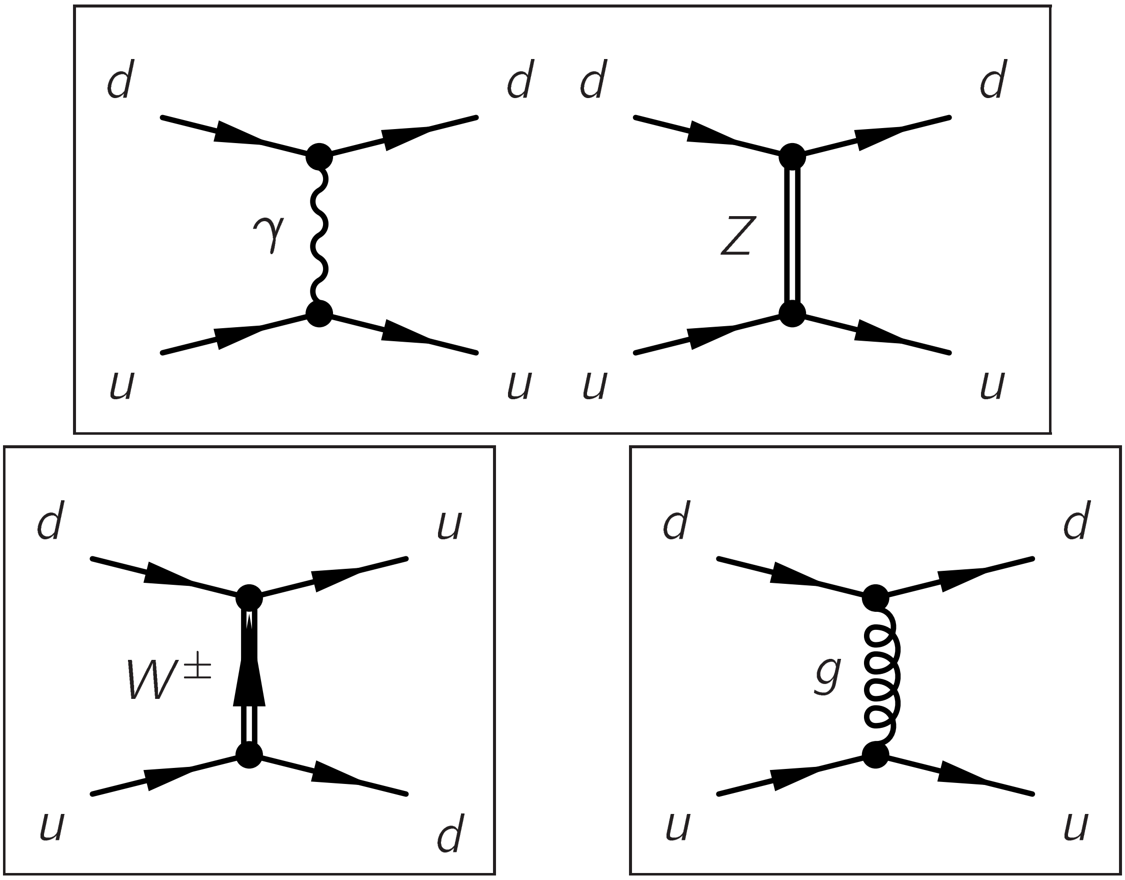




  ∵ $e^+e^- \to \mu^+\mu^-$ & $e^+\mu^- \to e^+\mu^-$ are also physical processes that give gauge invariant amplitudes

∴ conserved currents of (real or fictitious) horizontal (generation) symmetries can serve as a tool to separate gauge invariant classes of diagrams

- less trivial example: three separately gauge-invariant sets in $ud \to ud$

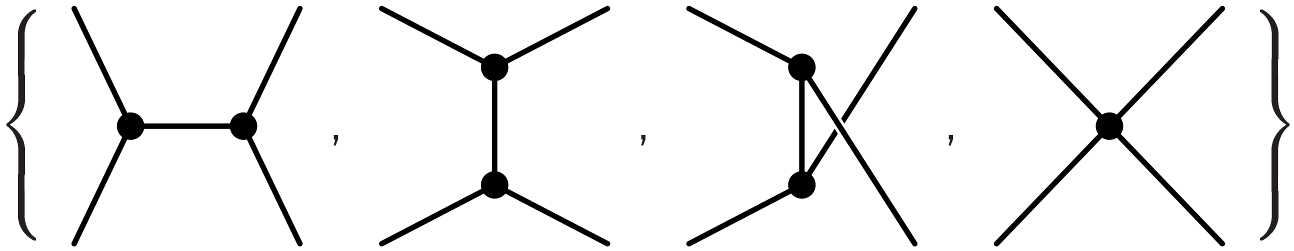


  $\because$ QCD is obvious, because the gauge group is a separate factor & we can switch off the coupling independently

  $\because$ charged current diagram is separate, because it is absent in $us \to us$ (assuming a diagonal CKM matrix).

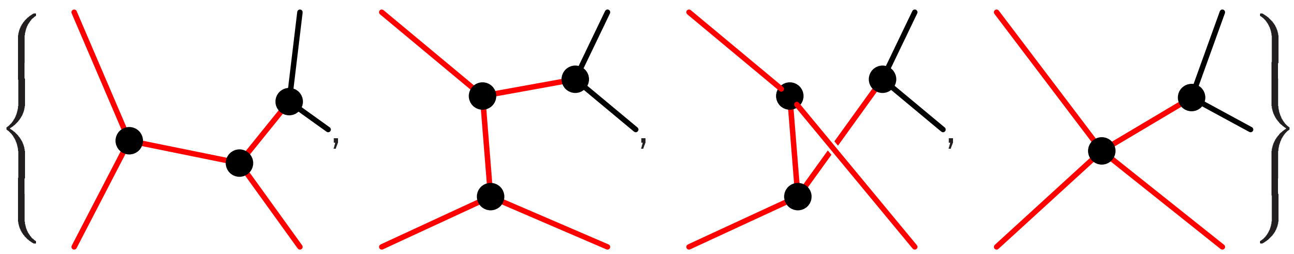
- … **useless unless we learn to deal with the combinatorics in more complicated applications** …

Consider the scalar four-point trees $T_4 = \{t_4^{(1)}, t_4^{(2)}, t_4^{(3)}, t_4^{(4)}\}$

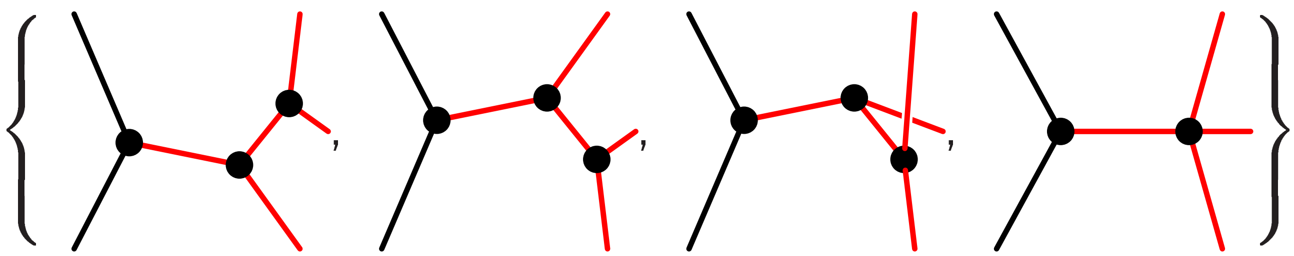


- $\because$ $t_4^{(1)}$, $t_4^{(2)}$, & $t_4^{(3)}$ have the same coupling from crossing (in gauge theories, $t_4^{(4)}$ is also fixed by symmetry), unless diagrams are forbidden by conserved quantum numbers.
- $\therefore$ flips, i. e. permutations of the set T_4, define a (trivial) symmetry of the four point amplitude.

More interesting: T_4 induces transformations in T_5:



- obviously, there is (in general) more than one $t_4 \in T_4$ embedded in a $t_5 \in T_5$



$\because$ there are 25 five-point trees in $\phi^3 + \phi^4$ & at most 6 can be reached from any diagram by a flip of a four-point subdiagram

☺ there must be a non-trivial mathematical structure . . .

Definition 1 *The flips in T_4 induce a relation $\circ$ on T_n:*

$$t \circ t' \Longleftrightarrow \exists t_4 \in T_4, t'_4 \in T_4 : t_4 \circ t'_4 \wedge t \setminus t_4 = t' \setminus t'_4$$

In words: $t \circ t'$ iff t' can be made from t by a **single** *flip of a four-point subdiagram*

NB: $\circ$ is **not** transitive & not an equivalence relation!

With the relation $\circ$, we can define a graph on the set T_n:

Definition 2 *The forest is a graph with each vertex representing a tree & the edges given by the flips of four-point subdiagrams*

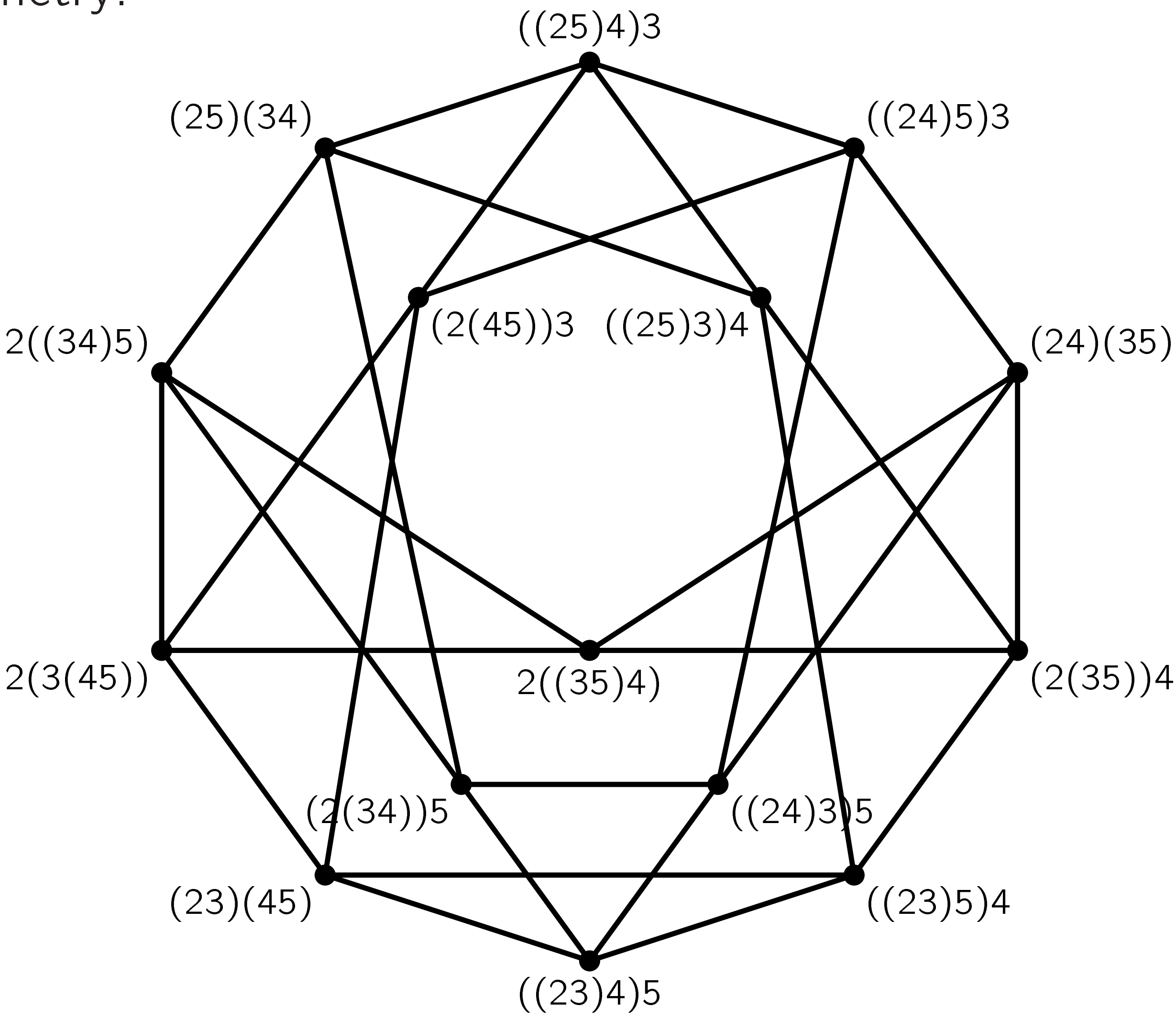
$$F(E) = \left\{ (t, t') \in T(E) \times T(E) \middle| t \circ t' \right\},$$

Theorem 1 *The unflavored forest $F(E)$ is connected for all external states E*

In other words: each diagram can be reached from any other by a succession of flips.

Proof 1 *Induction on the number of external particles*

Already the simplest example, the forest of the 15 five-point tree diagrams in unflavored ϕ^3-theory, has fascinating geometry:

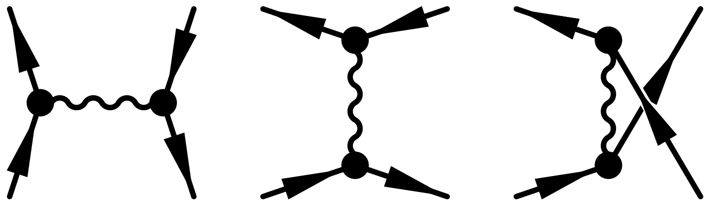


$|\,\mathbf{Aut}(F(E))| = 120$ permutations of the 15 vertices leave this graph invariant!

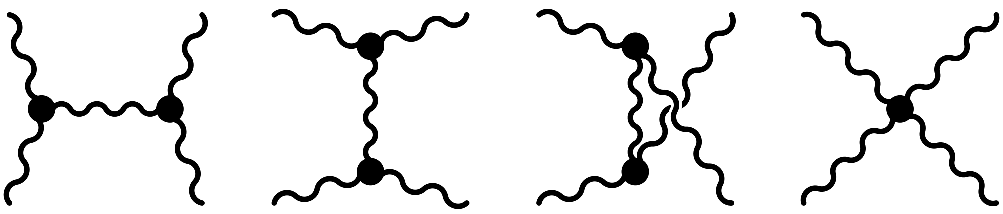
- ☺ probably an interesting mathematical problem
- ☹ probably no physical application, except . . .

Not all flips are created equal!

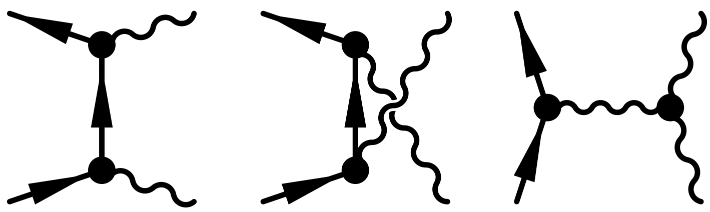
Four-point matter trees $\{t_4^{F,1}, t_4^{F,2}, t_4^{F,3}\}$ related by *flavor* flips:



Four-point gauge trees $\{t_4^{G,1}, t_4^{G,2}, t_4^{G,3}, t_4^{G,4}\}$ related by *gauge* flips:



Four-point "Compton" diagrams $\{t_4^{G,5}, t_4^{G,6}, t_4^{G,7}\}$ related by *gauge* flips (scalars have additional diagram):



- remember: *flavor* flips can be switched off by selection rules of (physical or artificial) horizontal ("generation") symmetry,
- *gauge* flips relate diagrams that **must** appear together in order to satisfy Ward identities.

∴ we have two different relations: a strong one & a weaker one:

$$t \bullet t' \Longleftrightarrow t \;\&\; t' \text{ related by } \textit{gauge} \text{ flip}$$
$$t \circ t' \Longleftrightarrow t \;\&\; t' \text{ related by } \textit{flavor or gauge} \text{ flip}$$

which define two different forests on the set of all tree diagrams for the external state E:

Definition 3

1. *the flavor forest (a. k. a. forest):*

$$F(E) = \left\{(t,t') \in T(E) \times T(E) \middle| t \circ t'\right\}$$

2. *the gauge forest*

$$G(E) = \left\{(t,t') \in T(E) \times T(E) \middle| t \bullet t'\right\} \subseteq F(E)$$

3. *the connected components $G_i(E)$ of $G(E)$:* groves

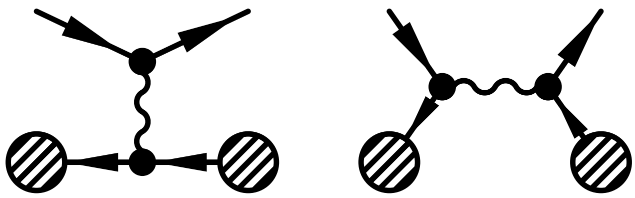
Theorem 2 *The forest $F(E)$ for an external state E consisting of gauge & matter fields is connected if the fields in E carry no conserved quantum numbers other than the gauge charges. The groves $G_i(E)$ are the minimal gauge invariant classes of Feynman diagrams.*

Proof 2

1. *The Ward identities require sums over* **all** *ways to attach a gauge boson to gauge charge carrying components of Feynman diagrams. Gauge flips connect pairs of neighboring insertions & can be iterated along gauge charge carrying propagators.*

$\therefore$ *no partition of the forest $F(E)$ that is finer than the groves $G_i(E)$ preserves gauge invariance.*

2. *There are two ways to add an additional pair of matter fields to a gauge invariant amplitude, which are related by a flavor flip:*



   (a) *If the flavor is new, the only way to attach the pair is through a gauge boson.*

   (b) *If the flavor is already present, we can also break up a matter field propagator.*

   Since it is always possible to introduce a new flavor, either physical or fictitious, without breaking gauge invariance, these cases fork off separately gauge invariant classes every time we add a new pair of matter fields.

Systematic procedure for calculating groves:

1. select the desired external state & simplify the combinatorics by treating all particles as outgoing

2. choose all quantum numbers identical except for gauge group quantum numbers (the flavor forest would otherwise be disconnected)

3. calculate the connected flavor forest & the groves

4. select the groves that are compatible with the non-gauge quantum numbers (flavor, etc.)

if only a few flavor combinations are interesting, it can be simpler in practical applications to generate all diagrams for the external states conventionally & to use the disconnected flavor forest.

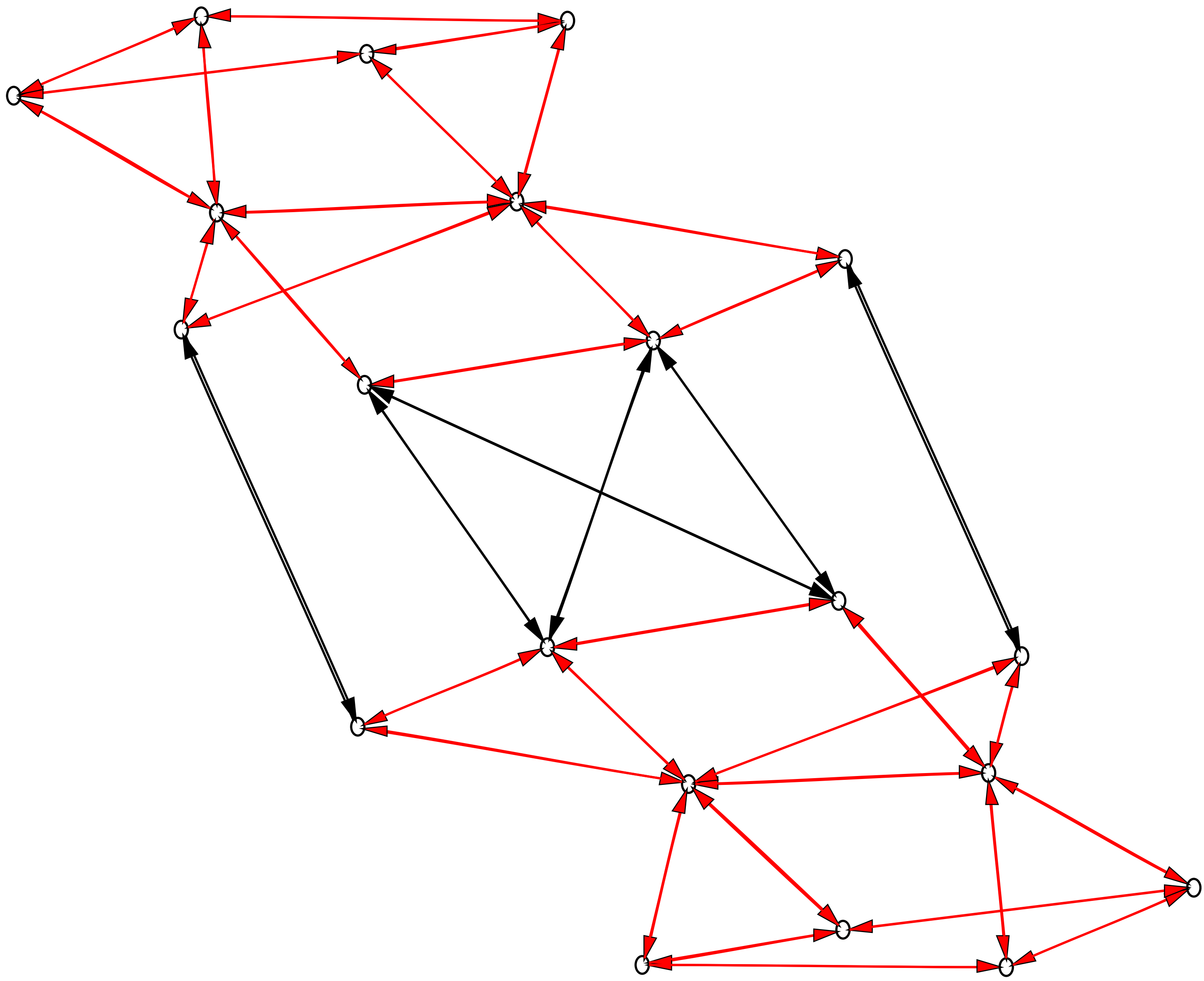
The resulting groves will be the same.

Groves for all processes with six massless fermions in the standard model (without QCD)

E	$\sum$	classes
$u\bar{u}u\bar{u}u\bar{u}$	144	$18 \cdot 8$
$u\bar{u}u\bar{u}u\bar{u}\gamma$	1008	$18 \cdot 24 + 36 \cdot 16$
$u\bar{u}u\bar{u}d\bar{d}$	92	$4 \cdot 11 + 6 \cdot 8$
$u\bar{u}u\bar{u}d\bar{d}\gamma$	716	$4 \cdot 95 + 6 \cdot 24 + 12 \cdot 16$
$\ell^+\ell^- u\bar{u}d\bar{d}$	35	$1 \cdot 11 + 3 \cdot 8$
$\ell^+\ell^- u\bar{u}d\bar{d}\gamma$	262	$1 \cdot 94 + 3 \cdot 24 + 6 \cdot 16$
$\ell^-\nu d\bar{u}d\bar{d}$	20	$2 \cdot 10$
$\ell^-\nu d\bar{u}d\bar{d}\gamma$	152	$2 \cdot 76$
$\ell^+\ell^-\ell^-\nu d\bar{u}$	20	$2 \cdot 10$
$\ell^+\ell^-\ell^-\nu d\bar{u}\gamma$	150	$2 \cdot 75$
$\ell^-\nu\ell^+\bar{\nu}d\bar{d}$	19	$1 \cdot 9 + 2 \cdot 4 + 1 \cdot 2$
$\ell^-\nu\ell^+\bar{\nu}d\bar{d}\gamma$	107	$1 \cdot 59 + 2 \cdot 12 + 2 \cdot 8 + 2 \cdot 4$
$\ell^-\bar{\nu}\ell^+\nu\ell^+\ell^-$	56	$4 \cdot 9 + 4 \cdot 4 + 2 \cdot 2$
$\ell^-\bar{\nu}\ell^+\nu\ell^+\ell^-\gamma$	328	$4 \cdot 58 + 4 \cdot 12 + 4 \cdot 8 + 4 \cdot 4$
$\ell^+\nu\ell^-\bar{\nu}\nu\bar{\nu}$	36	$4 \cdot 6 + 6 \cdot 2$
$\ell^+\nu\ell^-\bar{\nu}\nu\bar{\nu}\gamma$	132	$4 \cdot 26 + 2 \cdot 6 + 4 \cdot 4$
$\nu\bar{\nu}\nu\bar{\nu}\nu\bar{\nu}$	36	$18 \cdot 2$

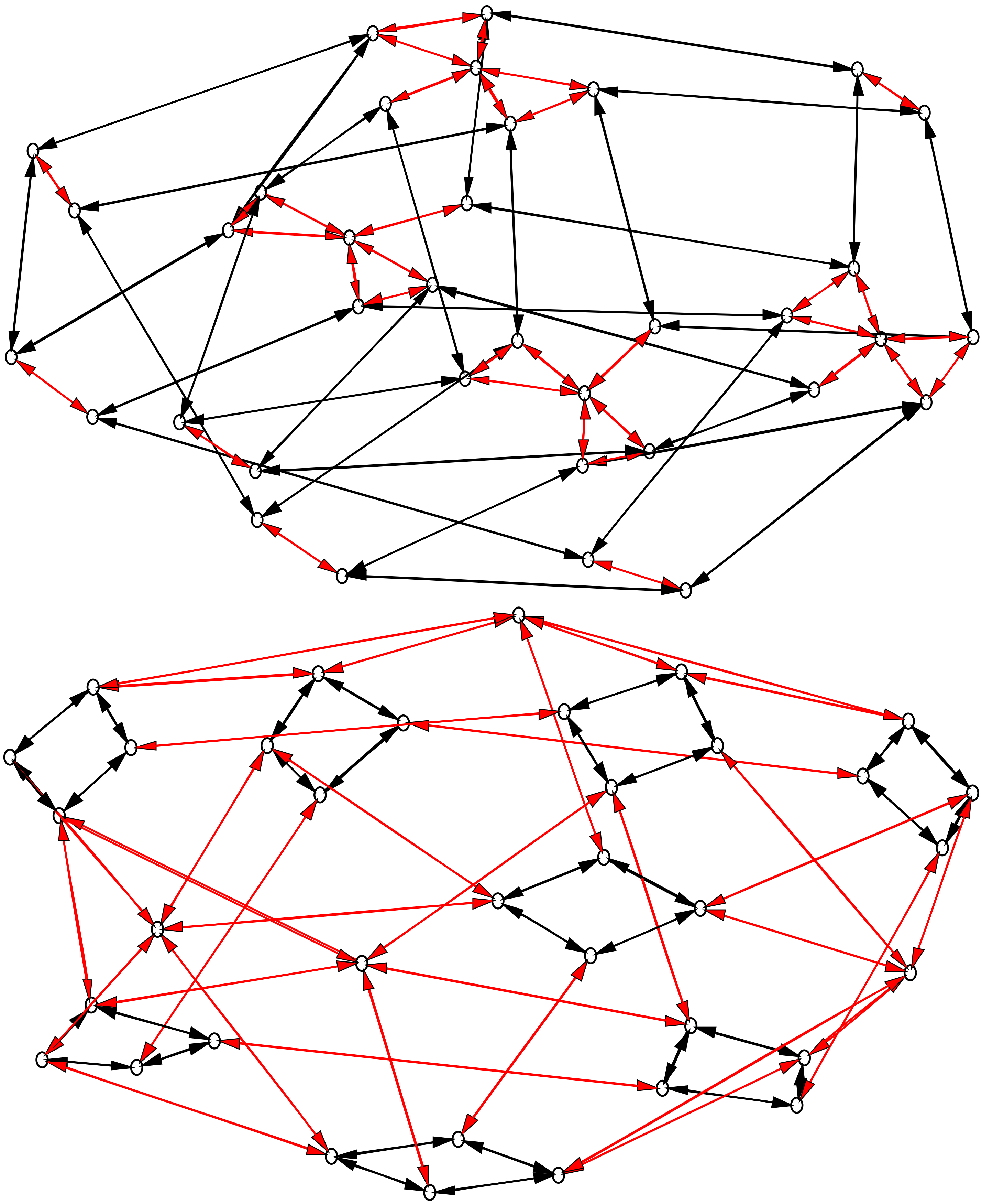
includes the familiar LEP2 classes (CC09, CC10, CC11, etc.)

As is well known, $e^+e^- \to e^-\bar{\nu}_e u\bar{d}$ falls into two separately gauge invariant classes, which look like

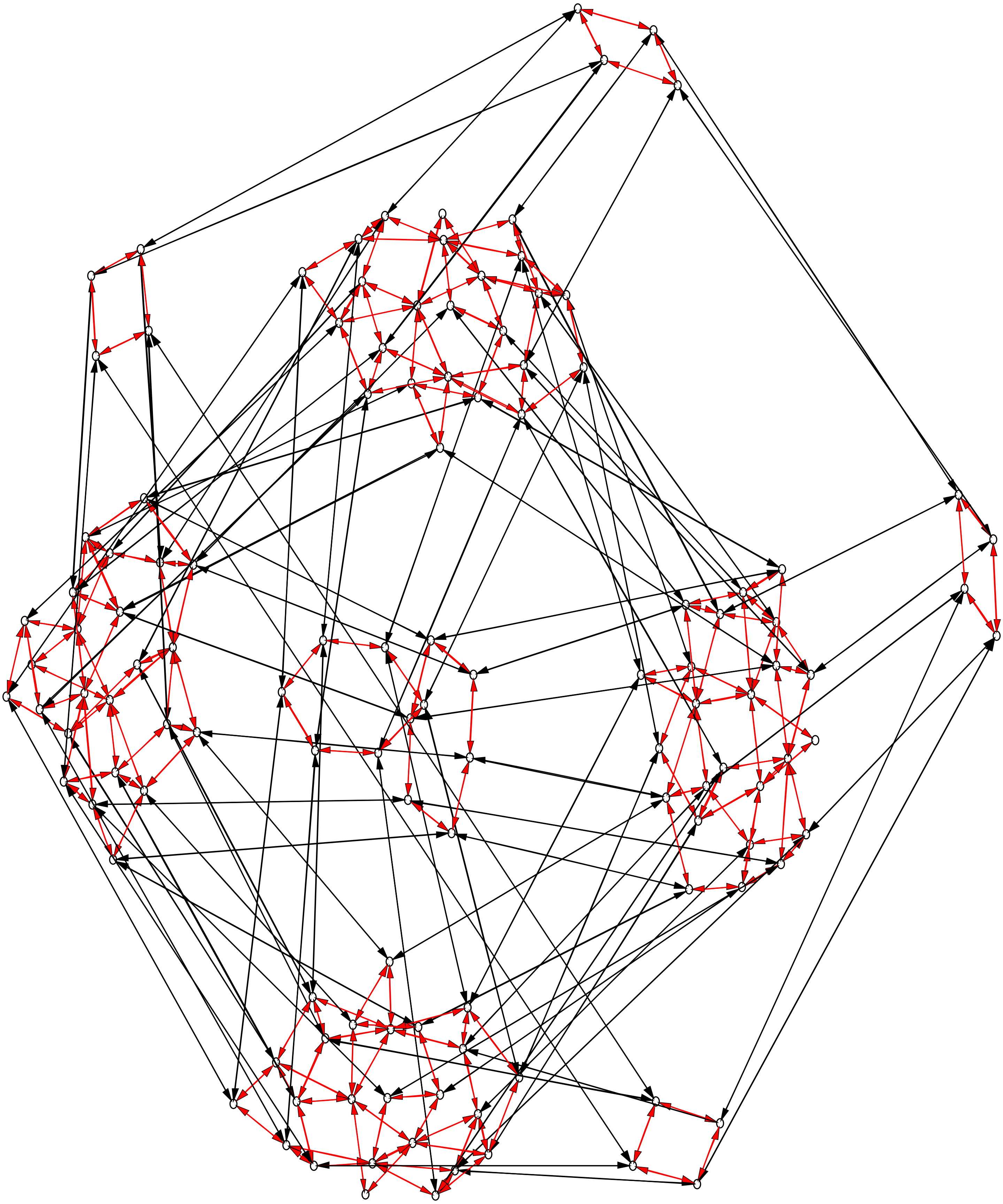


were gauge flips are red & flavor flips are black.

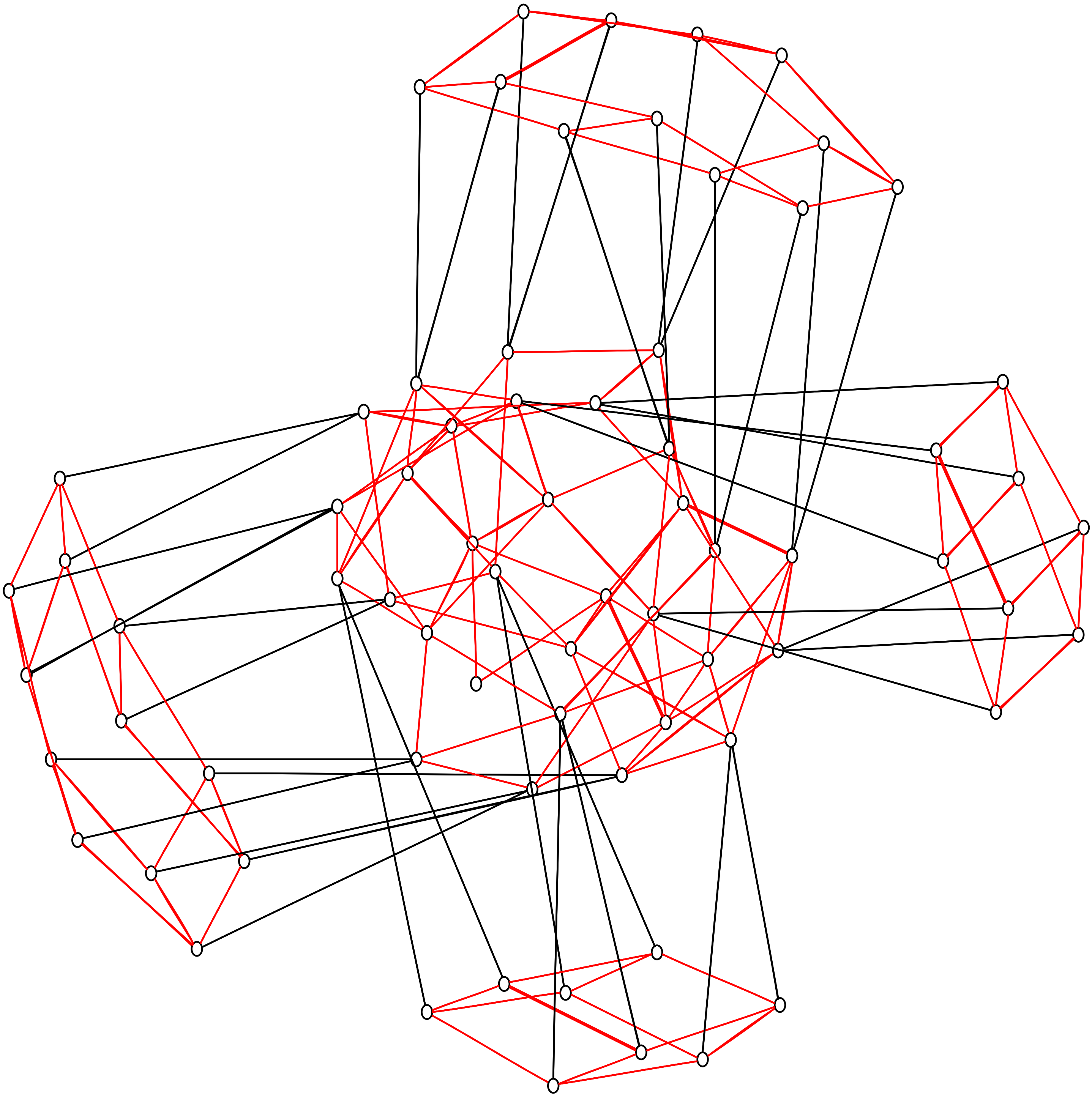
Two views of $e^+e^- \to \nu_e\bar{\nu}_e\nu_e\bar{\nu}_e$:



$e^+e^- \to \nu\bar{\nu}\nu\bar{\nu}\gamma$ is not the most pressing physics problem, but small enough to make a nice picture: $4 \cdot 26 + 2 \cdot 6 + 4 \cdot 4 \Longleftrightarrow$



The forest of size 71 for the process $\gamma\gamma \to u\bar{d}d\bar{u}$ in the standard model (without QCD, CKM mixing & Higgs contributions) with one grove of size 31, two groves of size 12 & two groves of size 8.



Surprise: $|\mathbf{Aut}(F(E))| = 128$