Disruption, anti-pinch, and effective Υ in e^-e^- linear colliders

K. A. Thompson & Pisin Chen
SLAC

We present an analytic approximation for

$$H_D = \frac{\mathcal{L}}{\mathcal{L}_0}$$

for e^-e^- collisions, and verify that the standard formulas for Υ_{eff} and n_γ used for e^+e^- also work for e^-e^-.

We focus on the case of round, Gaussian beams.

The geometric luminosity per bunch, not taking account of disruption or hour-glass effect, is given by

$$\mathcal{L}_0 \equiv \frac{N^2}{4\pi\sigma_0^2} \quad , \tag{1}$$

where N is the number of particles per bunch and σ_0 is the transverse beam size. We assume the beam distributions are Gaussian longitudinally and transversely.

The hour-glass effect reduces the undisrupted luminosity unless the parameter

$$A \equiv \frac{\sigma_z}{\beta^*} \tag{2}$$

is much less than 1. Here σ_z is the bunch length and β^* is the betatron function at the collision point.

The disruption parameter D

$$D = \frac{r_e \sigma_z N}{\gamma \sigma_0^2} \quad . \tag{3}$$

We define the luminosity pinch enhancement factor by

$$H_D \equiv \frac{\mathcal{L}_D}{\mathcal{L}_0} \quad . \tag{4}$$

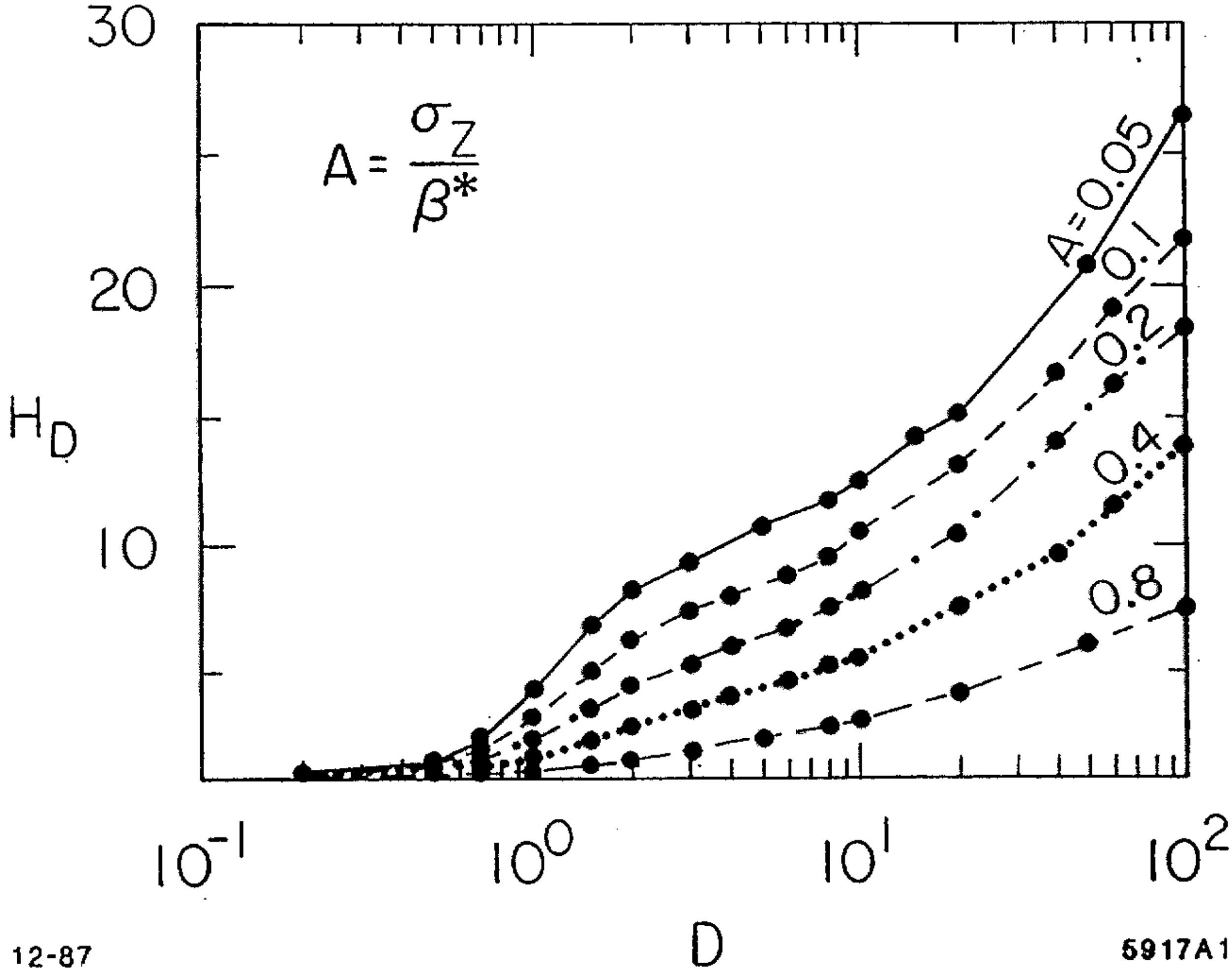
Here $\mathcal{L}_D$ denotes the actual luminosity with disruption and hour-glass effect taken into account. (Caution: H_D is sometimes defined as $\mathcal{L}_D/\mathcal{L}_A$ where $\mathcal{L}_A$ is the luminosity with hour-glass effect taken into account.) For e^-e^- collisions, the "enhancement" factor is of course less than 1. For e^+e^- collisions, H_D is greater than one, and if the beams are round the approximate analytic formula given in Reference [1] for the luminosity pinch enhancement factor is

(Yokoya & Chen)

$$H_D \approx 1 + D^{1/4}\left(\frac{D^3}{1+D^3}\right)\left[\ln(\sqrt{D}+1) + 2\ln\frac{0.8}{A}\right] \tag{5}$$

This is an empirical formula obtained by fitting simulation results [4], and is good to about 10% over reasonable ranges of the parameters A and D.

e^+e^- round beams



[P. Chen + K. Yokoya
Phys Rev D38 987 (1988)]

TABLE 1. e-e- IP parameters for round-beam (b) design in Zimmermann,et.al.*

E_{beam} [GeV]	500.
N [10^{10}]	0.95
$\gamma\epsilon$ [10^{-6} m-rad]	1.
β^* [mm]	0.25
σ_z [μm]	125.
σ_0 [nm]	16.0
$\mathcal{L}_0$ [10^{33} m^{-2}]	28.12
A	0.500
D	13.4

* Proceedings of e^-e^- Workshop, Santa Cruz, Calif, 22-24 Sept 1997.

Simulations using GUINEAPIG and CAIN give $H_D \approx 0.25$ for this design:

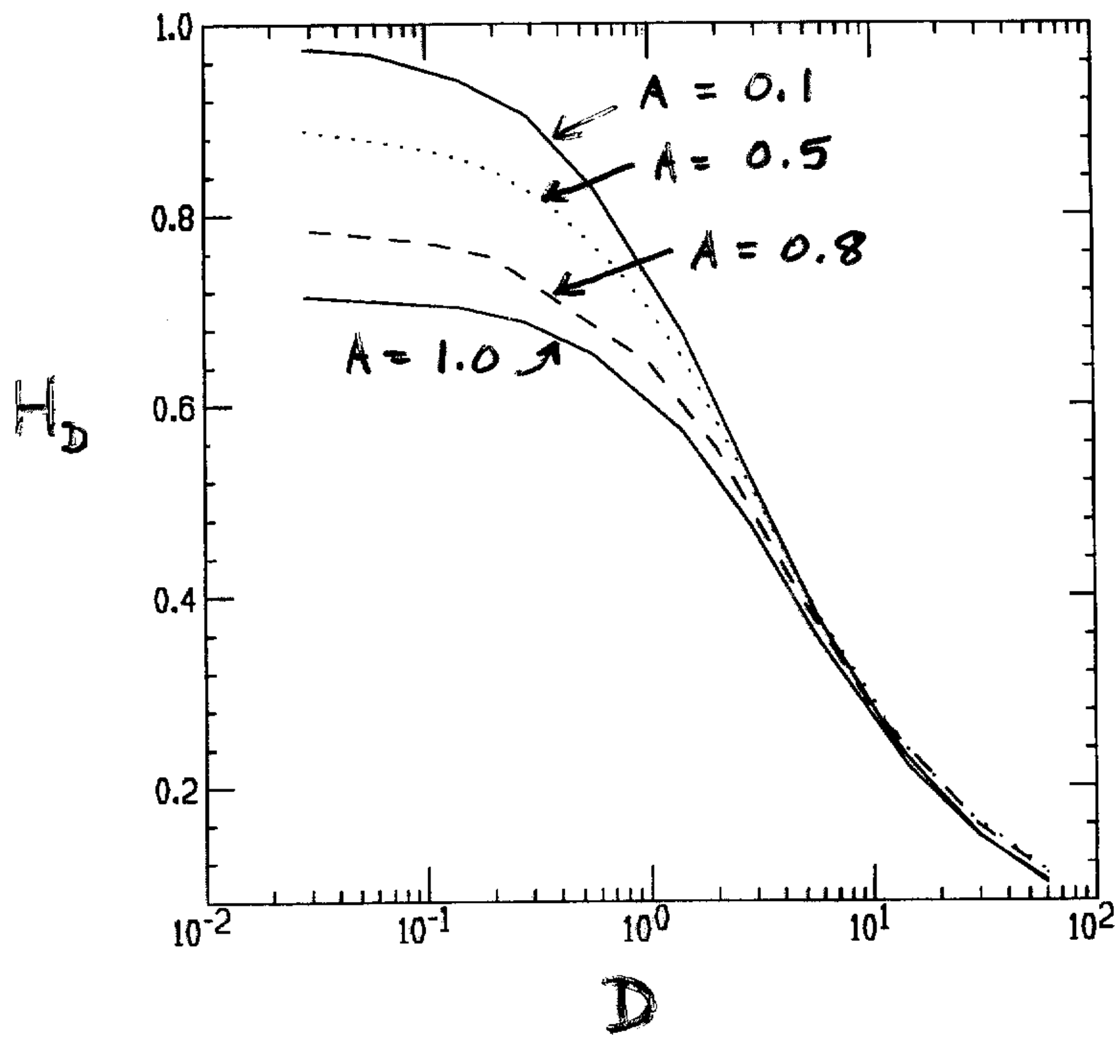
TABLE 2. Luminosity simulation results with and without beamstrahlung turned on

	Guineapig (disruption and beamstrahlung)	Guineapig (disruption, no beamstrahlung)	CAIN (disruption and beamstrahlung)	CAIN (disruption, no beamstrahlung)
$\mathcal{L}_D$ [10^{33} m^{-2}]	6.9	7.3	6.7	7.2
$H_D \equiv \mathcal{L}_D/\mathcal{L}_0$ (sim)	0.25	0.26	0.24	0.26

e^-e^-

GUINEAPIG (D. Schulte's beam-beam program)

simulation results



The major features of these curves can be easily understood on physical grounds. For very small disruption, H_D asymptotically approaches the value expected from the hour-glass effect alone:

$$H_D \approx \eta_A \equiv \frac{2}{\sqrt{\pi}A}\int_0^\infty \frac{e^{-u^2/A^2}}{1+A^2u^2}du \quad , \tag{6}$$

For $0 < A < 1$, a reasonably good expansion is $\eta_A \approx 1 - A^2/4$.

One might try to factorize H_D as $H_D = F(D)\eta_A$. Note, however, that for very large disruption the divergence of the beam due to the final focus system, represented by A, will be completely overwhelmed, explaining why the simulation curves for different A converge at large D. For $D >> 1$, the beams disrupt each other away within a distance σ_z/D and the effective value of A becomes

$$\tilde{A} = A/D = \frac{2(\gamma\epsilon)}{r_e N} \quad , \tag{7}$$

depending only on inherent properties of the beam.

To smooth the transition between the regimes of D, we take

$$\eta(A, D) \approx 1 - \frac{1}{4}\left(\frac{A}{1+bD}\right)^2 \tag{8}$$

where b is an adjustable parameter (we can in addition adjust the parameter 1/4 in front).

A derivation of F_D for round, Gaussian beams and small D [4] goes through for e^-e^- with only a change of sign from that for e^+e^-, and yields

$$F_D \approx 1 - \frac{2D}{3\sqrt{\pi}} \quad . \tag{9}$$

This is just the small-argument expansion of $\exp -\frac{2D}{3\sqrt{\pi}}$ so we might try matching onto that for larger D. One finds that the exponential drops off too quickly, but a modified Bessel function I_0 can be introduced to moderate this drop-off. We try

$$F_D = \exp -\frac{2D}{3\sqrt{\pi}} I_0(\delta D) \quad , \tag{10}$$

where δ is another adjustable parameter.

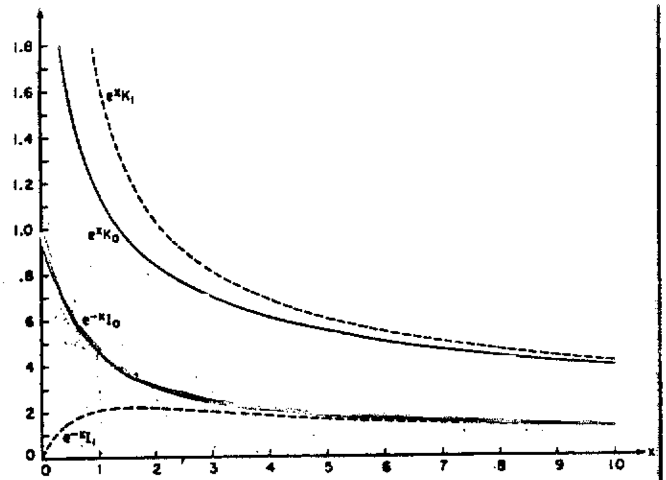


FIGURE 9.8. $e^{-x}I_0(x), e^{-x}I_1(x), e^xK_0(x)$ and $e^xK_1(x)$.

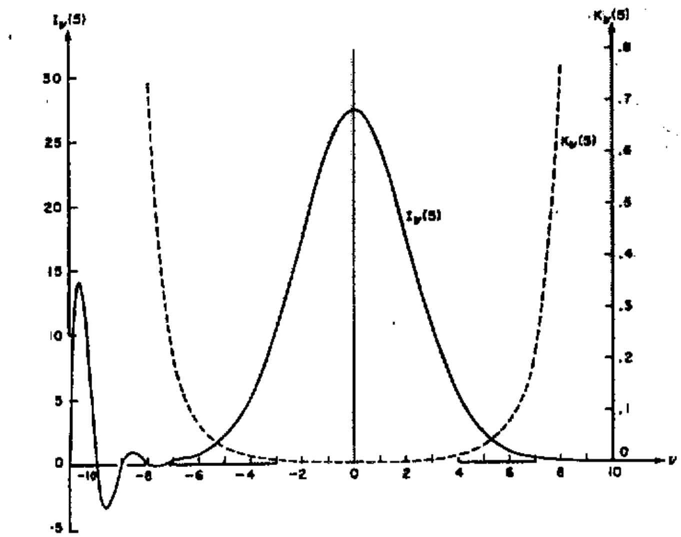


FIGURE 9.9. $I_\nu(5)$ and $K_\nu(5)$.

Relations Between Solutions

9.6.2 $$K_\nu(z)=\tfrac{1}{2}\pi\frac{I_{-\nu}(z)-I_\nu(z)}{\sin(\nu\pi)}$$

The right of this equation is replaced by its limiting value if ν is an integer or zero.

9.6.3

$$I_\nu(z)=e^{-\frac{1}{2}\nu\pi i}J_\nu(ze^{\frac{1}{2}\pi i}) \qquad (-\pi<\arg z\leq\tfrac{1}{2}\pi)$$

$$I_\nu(z)=e^{3\nu\pi i/2}J_\nu(ze^{-3\pi i/2}) \qquad (\tfrac{1}{2}\pi<\arg z\leq\pi)$$

9.6.4

$$K_\nu(z)=\tfrac{1}{2}\pi i e^{\frac{1}{2}\nu\pi i}H_\nu^{(1)}(ze^{\frac{1}{2}\pi i}) \qquad (-\pi<\arg z\leq\tfrac{1}{2}\pi)$$

$$K_\nu(z)=-\tfrac{1}{2}\pi i e^{-\frac{1}{2}\nu\pi i}H_\nu^{(2)}(ze^{-\frac{1}{2}\pi i}) \qquad (-\tfrac{1}{2}\pi<\arg z\leq\pi)$$

9.6.5

$$Y_\nu(ze^{\frac{1}{2}\pi i})=e^{\frac{1}{2}(\nu+1)\pi i}I_\nu(z)-(2/\pi)e^{-\frac{1}{2}\nu\pi i}K_\nu(z) \qquad (-\pi<\arg z\leq\tfrac{1}{2}\pi)$$

9.6.6 $$I_{-n}(z)=I_n(z),\ K_{-\nu}(z)=K_\nu(z)$$

Most of the properties of modified Bessel functions can be deduced immediately from those of ordinary Bessel functions by application of these relations.

Limiting Forms for Small Arguments

When ν is fixed and $z\rightarrow0$

9.6.7

$$I_\nu(z)\sim(\tfrac{1}{2}z)^\nu/\Gamma(\nu+1) \qquad (\nu\neq-1,-2,\ldots)$$

9.6.8 $$K_0(z)\sim-\ln z$$

9.6.9 $$K_\nu(z)\sim\tfrac{1}{2}\Gamma(\nu)(\tfrac{1}{2}z)^{-\nu} \qquad (\mathscr{R}\nu>0)$$

Ascending Series

9.6.10 $$I_\nu(z)=(\tfrac{1}{2}z)^\nu\sum_{k=0}^{\infty}\frac{(\frac{1}{4}z^2)^k}{k!\Gamma(\nu+k+1)}$$

9.6.11

$$K_n(z)=\tfrac{1}{2}(\tfrac{1}{2}z)^{-n}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}(-\tfrac{1}{4}z^2)^k$$
$$+(-)^{n+1}\ln(\tfrac{1}{2}z)I_n(z)$$
$$+(-)^n\tfrac{1}{2}(\tfrac{1}{2}z)^n\sum_{k=0}^{\infty}\{\psi(k+1)+\psi(n+k+1)\}\frac{(\frac{1}{4}z^2)^k}{k!(n+k)!}$$

where $\psi(n)$ is given by 6.3.2.

9.6.12 $$I_0(z)=1+\frac{\frac{1}{4}z^2}{(1!)^2}+\frac{(\frac{1}{4}z^2)^2}{(2!)^2}+\frac{(\frac{1}{4}z^2)^3}{(3!)^2}+\ldots$$

9.6.13

$$K_0(z)=-\{\ln(\tfrac{1}{2}z)+\gamma\}I_0(z)+\frac{\frac{1}{4}z^2}{(1!)^2}$$
$$+(1+\tfrac{1}{2})\frac{(\frac{1}{4}z^2)^2}{(2!)^2}+(1+\tfrac{1}{2}+\tfrac{1}{3})\frac{(\frac{1}{4}z^2)^3}{(3!)^2}+\ldots$$

Wronskians

9.6.14

$$W\{I_\nu(z),I_{-\nu}(z)\}=I_\nu(z)I_{-(\nu+1)}(z)-I_{\nu+1}(z)I_{-\nu}(z)$$
$$=-2\sin(\nu\pi)/(\pi z)$$

9.6.15

$$W\{K_\nu(z),I_\nu(z)\}=I_\nu(z)K_{\nu+1}(z)+I_{\nu+1}(z)K_\nu(z)=1/z$$

Analytic approximation

$\delta = 0.376$

$$\boxed{H_D = \left[1 - 0.29\frac{A^2}{(1+0.4D)^2}\right] e^{-\frac{2D}{3\sqrt{\pi}}} I_0(\delta D)\, f_{choc}} \tag{11}$$

The modified Bessel function has expansions for large and small D that agree well for $D \sim 1$ and are given by:

$$I_0(\alpha D) = \begin{cases} 1 + \dfrac{(\delta D)^2}{4} + \dfrac{(\delta D)^4}{32} + \dfrac{(\delta D)^6}{2304} & (\delta D < 1) \\[2ex] \dfrac{e^{\delta D}}{\sqrt{2\pi\delta D}}\left[1 + \dfrac{1}{8\delta D} + \dfrac{9}{128(\delta D)^2}\right] & (\delta D > 1) \quad . \end{cases} \tag{12}$$

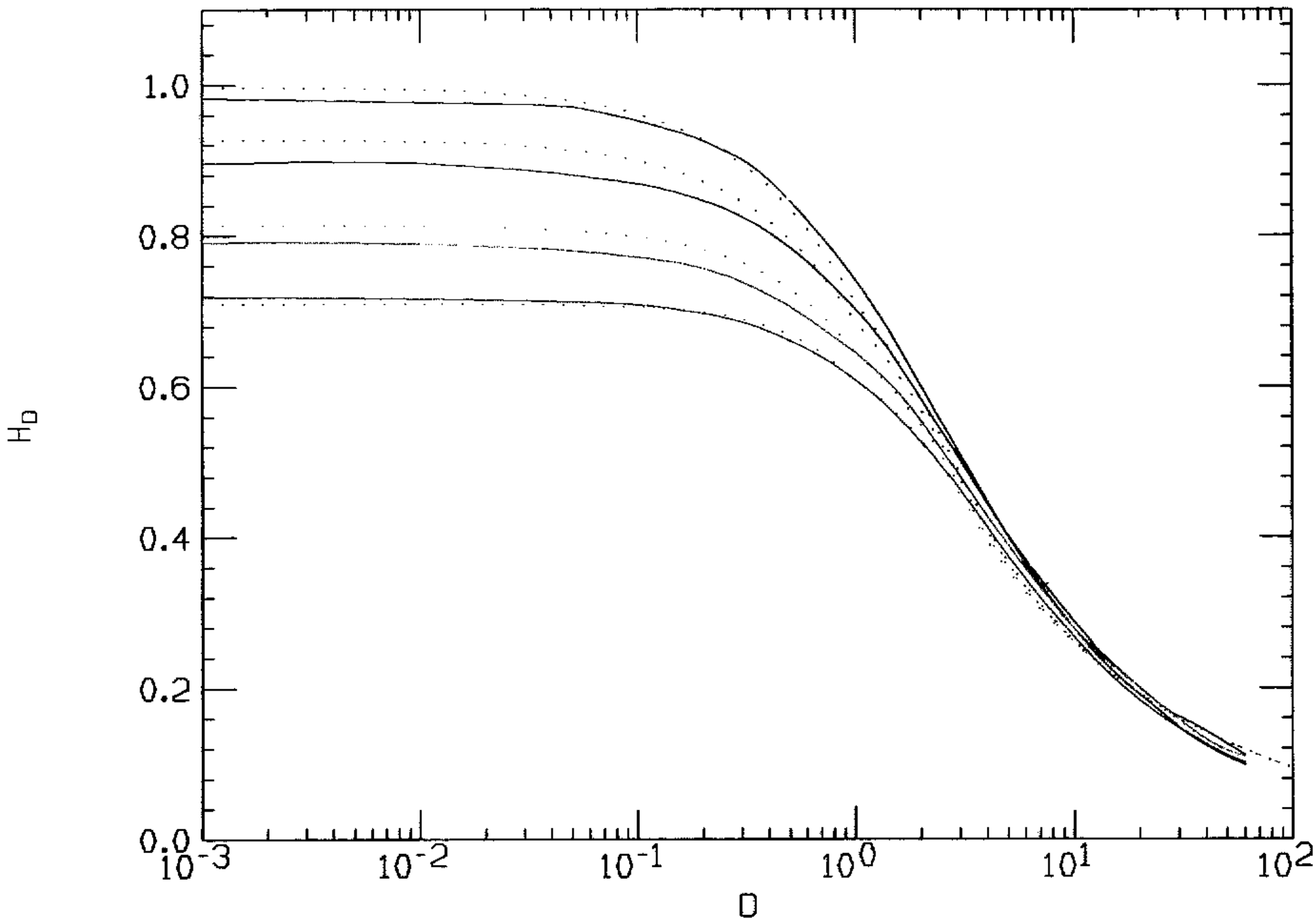
(13)

We have also introduced a purely empirical fudge factor f_{choc} to further improve the fit to simulation results:

$$f_{choc} = \begin{cases} 1 & (D < 1) \\[2ex] 1 + 0.1 \ln D & (D > 1) \quad . \end{cases} \tag{14}$$

(15)

Good to better than 5% for all interesting A, D



Solid curves : GUINEAPIG simulations
Dotted curves : Analytic approximation

Top to bottom:
A = 0.1
A = 0.5
A = 0.8
A = 1.0

BEAMSTRAHLUNG AND APPROXIMATION OF Υ

The strong electromagnetic field of the oncoming beam not only affects the luminosity through the effects of disruption, it also causes particles to radiate beamstrahlung photons as they are bent by the strong field. Consider an electron or positron of very high energy E traversing a strong electromagnetic field. Such a situation may be characterized by the Lorentz invariant parameter Υ, defined by

$$\Upsilon \equiv \frac{e\hbar}{m^3c^4}\sqrt{|F_{\mu\nu}p^\nu|^2} = \gamma\frac{B}{B_c} \tag{16}$$

Here $p^\nu = (E, \vec{p})$ is the 4-momentum of the incoming electron or positron, m is the electron mass, $\gamma \equiv E/mc^2$ is the usual Lorentz factor, $F_{\mu\nu}$ is the energy-momentum tensor of the electromagnetic field, $B = |\vec{B}| + |\vec{E}|$, and $B_c \equiv m^2c^3/\hbar e \approx 4.4 \times 10^{13}$ Gauss is the Schwinger critical field.

Yokoya and Chen also give analytic approximations for n_γ, the average number of beamstrahlung photons produced per incoming beam particle:

$$\boxed{n_\gamma \approx 1.06\frac{\alpha N r_e}{\bar{\sigma}}\frac{1}{(1+\Upsilon_{eff}^{2/3})^{1/2}} \quad ,} \tag{17}$$

and δ_B, the average beamstrahlung energy loss per particle:

$$\delta_B \approx 0.216\frac{r_e^3N^2\gamma}{\sigma_z\bar{\sigma}^2}\frac{1}{[1+(1.5\Upsilon_{eff})^{2/3}]^2} \quad . \tag{18}$$

However, the equation for δ_B does not take account of multiple photon emissions, so is of limited use for our purposes here.

The effective average beamstrahlung parameter Υ, in the analytic approximation of Yokoya and Chen [1] for round, Gaussian beams and small D is:

$$\Upsilon_{eff} = \frac{5}{12}\Upsilon_{max} = \frac{5Nr_e^2\gamma}{12\alpha\sigma_z\bar{\sigma}} \tag{19}$$

Here $\bar{\sigma}$ is the effective beam size during the collision. If the disruption is not too strong this is not very different from the nominal transverse beam size $\sigma_0 (= \sigma_x = \sigma_y$ for round beams). If the disruption is significant, one may calculate an effective beam size during the collision [6] from

$$\bar{\sigma} = H_D^{-1/2} \sigma_0 \quad . \tag{20}$$

We may then use this effective beam size to calculate Υ_{eff} and n_γ using the above equations.

This works in e^-e^- as well as in e^+e^-, as is borne out by our simulations. For example, using the parameters in Table 1, we ran Guineapig with beamstrahlung turned on, in order to calculate n_γ and H_D. We modified Guineapig to allow us to calculate beamstrahlung with disruption turned off. The results with and without disruption are shown in Table 3.

When disruption is turned off, $\bar{\sigma} \approx \sigma_0$=16 nm. From the above formulas, we then get $\Upsilon_{eff} \approx 2.1$, so that we expect $n_\gamma \approx 8$, in good agreement with the simulation result of 8.7.

When disruption is turned on, $H_D \approx 0.25$ from either the simulation or our analytic approximation of the previous section. Thus $\bar{\sigma} \approx 32$ nm. From the above formulas, we then get $\Upsilon_{eff} \approx 1.05$, so that we expect $n_\gamma \approx 4.5$, again in good agreement with the simulation result of 4.4.

	Guineapig (beamstrahlung and disruption)	Guineapig (beamstrahlung, no disruption)
$\mathcal{L}_D$ [10^{33} m^{-2}]	6.9	25.1
$H_D \equiv \mathcal{L}_D/\mathcal{L}_0$ (sim)	0.25	0.89
n_γ	4.2	8.7
δ_B	39%	62%

Summary

For round e^-e^- beams,

$$H_D \approx \left[1 - 0.29 \frac{A^2}{(1+0.4D)^2}\right] e^{-2D/3\sqrt{\pi}} \cdot I_0(0.376\,D) \cdot \left[1 + 0.1 \ln D \cdot h(D-1)\right]$$

where

$$h(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

and

$$I_0(x) \approx \begin{cases} 1 + \dfrac{x^2}{4} + \dfrac{x^4}{32} + \dfrac{x^6}{2304} & x < 1 \\ \dfrac{e^x}{\sqrt{2\pi x}}\left[1 + \dfrac{1}{8x} + \dfrac{9}{128x^2}\right] & x > 1 \end{cases}$$

Just as for round e^+e^- beams, we can use:

$$\Upsilon_{eff} = \frac{5 N r_e^2 \gamma}{12 \alpha \sigma_z \bar{\sigma}}$$

where $\bar{\sigma} = H_D^{-1/2} \sigma_0$ is effective beam size.

$$n_\gamma \approx 1.06 \frac{\alpha N r_e}{\bar{\sigma}} \frac{1}{(1 + \Upsilon_{eff}^{2/3})^{1/2}}$$