An introduction to rectangular Cavity Beam Position Monitors

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Contents

1 Introduction

The purpose of this text is to explain how Beam Position Monitors (BPMs) work, especially rectangular ones. Most of this text may be redundant with what you can find in [1] or in [2] but they only develop the case of circular cavities which are the most commonly used.
We propose here to go back from the start and to focus on rectangular cavities. We will begin with some electromagnetism equations and then adopt a more experimental point of view for the study of cavity BPMs.

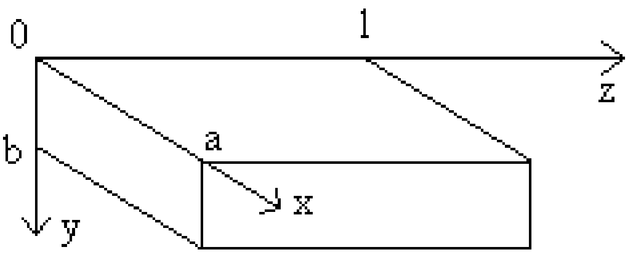
2 Theoretical elements for cavity BPMs

2.1 Electric field in a rectangular waveguide or cavity

We consider an electric field (E-field) propagating along the z axis in a waveguide ($x \in [0, a]; y \in [0, b]$) as seen in the following picture (and $z \in [0, l]$ if in a cavity). The E-field obeys the equation :

$$\Delta \overrightarrow{E} = \frac{1}{c^2} \frac{\partial^2 \overrightarrow{E}}{\partial t^2} \tag{1}$$

where c is the speed of light (we assume that the middle is vacuum).



Rectangular cavity with axis x,y and z

If we assume that $\overrightarrow{E}\ \alpha\ \exp(i\omega t)$ (what is always the case), we get,

$$\Delta \overrightarrow{E} = -k^2 \overrightarrow{E} \quad \text{using} \quad k = \frac{\omega}{c} \tag{2}$$

where k is the wavenumber of the E-field. It is then easy to separate variables, writing $k^2 = k_x^2 + k_y^2 + k_z^2$.
Nota Bene : with different cavities or conditions, it is often better to write

$k^2 = k_c^2 + k_z^2$, where k_c is called the cutting wavenumber. See [1] or [2] for more detailed calculations and examples.
Now, it is important to notice that we are only interested in the z axis component of the E-field.This gives the following equation (after simplification by $\exp(i\omega t)$) :

$$\Delta E_z = -(k_x^2 + k_y^2 + k_z^2)E_z \tag{3}$$

Then, the solution for E_z can be written using the form :
$E_z = f(x)g(y)h(z)$ and we can separate the previous equation to the following equations :

$$\frac{d^2 f}{dx^2} = -k_x^2 f \quad \frac{d^2 g}{dy^2} = -k_y^2 g \quad \frac{d^2 h}{dz^2} = -k_z^2 h \tag{4}$$

There are now two possibilities :

- in a waveguide : we assume that the E-field is propagating forward. This gives : $h(z) = H_0 \exp(-ik_z z)$.
- in a cavity : the wave is a standing wave and we can apply the method that we will use for the x and y axis too.

There are boundary conditions on x and y. Indeed, the E-field must be equal to 0 on the borders of the cavity. We will only solve the equation 4 for the x case because the y one would be the same. We get the standing wave solution : $f(x) = A\cos(k_x x) + B\sin(k_x x)$ with the conditions $f(0) = 0$ and $f(a) = 0$ which implies $f(x) = F_0 \sin(k_x x)$ where k_x is quantified by the following expression :

$$k_x = \frac{m\pi}{a} \quad (m \in \mathbb{Z}) \tag{5}$$

Then, we can deduce the wavelength λ defined by :

$$\lambda = \frac{2\pi}{k} \tag{6}$$

and also the frequency f defined by :

$$f = \frac{\omega}{2\pi} = \frac{kc}{2\pi} = \frac{c}{\lambda} \tag{7}$$

We have now the solution of the first equation in the two cases :

- in a waveguide : $E_z = E_0 \sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})\exp(-ik_z z)$ and we can write the wavelength and the frequency :

$$\lambda = \frac{2}{\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2 + k_z^2}} \tag{8}$$

$$f = \frac{c}{2}\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2 + k_z^2} \tag{9}$$

- in a cavity : $E_z = E_0 \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) \sin(\frac{q\pi z}{l})$ and the corresponding wavelength and frequency :

$$\lambda_{mnq} = \frac{2}{\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2 + (\frac{q}{l})^2}} \tag{10}$$

$$f_{mnq} = \frac{c}{2}\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2 + (\frac{q}{l})^2} \tag{11}$$

2.2 About TM modes

We are now using the previous results for cavities. The transverse magnetic modes (TM) (ie where $H_z = 0$) can be easily identified using their frequencies given by the formula 11. In our case, q will always be equal to 0. Indeed, l can be 10 times inferior to a and b. This makes $f_{mn1} \sim 10 f_{mn0}$, what is more difficult to analyze since $f_{mn0} \sim 5GHz$. And we are only interested in the x and y modes.
Besides, even if $q = 0$, *the E_z term is not equal to zero* for our boundary conditions have changed : the cavity is opened so the sine function changes by a phase factor that makes it different from zero.
Moreover, since the sine would be equal to 0 if we assume $m = 0$ or $n = 0$, we will use the following definition of TM modes : $TM_{ijk} = TM_{m-1,n-1,q}$. For example, TM_{100} means $m = 2$, $n = 1$ and $q = 0$. This definition is consistent with the common rule that defines i and j as the number of nodes of the field in the cavity.
The purpose of our Cavity BPM is to measure the beam offset and the beam slope. Thus we are interested in the dependance between the TM modes and the beam deviation.
Nota Bene : we center now the z axis in the middle of the cavity, $z \in [-l/2, l/2]$.

2.3 Beam offset

The parameter that we will be able to measure is the output voltage in the cavity. There is a linear dependance between the output voltage and the voltage of the excited mode that we will see later. The voltage of the mode ij is simply given by :

$$V_{ij} = \int_{-l/2}^{l/2} \overrightarrow{E}.\overrightarrow{ds} \tag{12}$$

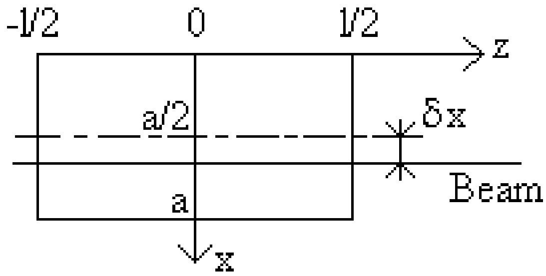
Replacing in this equation $\overrightarrow{E}$ by $E_0 \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) \exp(i\omega t)\overrightarrow{u_z}$, where we must change variables from t to z using $\omega t = kz$, we get the following equation :

$$V_{ij} = \int_{-l/2}^{l/2} E_0 \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) \exp(ikz) dz \tag{13}$$

And an integration over z of $\exp(ikz)$ gives :

$$V_{ij} = E_0 sinc(\frac{kl}{2}) l \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) \tag{14}$$

The factor $sinc(\frac{kl}{2}) = \frac{\sin(\frac{kl}{2})}{\frac{kl}{2}}$ is often named T_{tr}.



Notice : We have chosen coordinates where the center of the cavity is $(a/2, b/2)$ (see picture above). So we must develop $\sin(\frac{m\pi x}{a})$ around $a/2$. We get : $\sin(\frac{m\pi}{2}) + (\frac{m\pi\delta x}{a})\cos(\frac{m\pi}{2})$. This means that if m is odd, this factor does not depend on δx the beam offset. However, if m is even, we get a signal proportional to δx.

Therefore, the most important modes are the TM_{10} and TM_{01} to measure the beam offset. They are called *dipole modes*. We may also calculate the voltages given by the first modes :

$$V_{00} = E_0 T_{tr} l \tag{15}$$

$$V_{10} = E_0 T_{tr} l \frac{2\pi\delta x}{a} \tag{16}$$

$$V_{01} = E_0 T_{tr} l \frac{2\pi\delta y}{b} \tag{17}$$

$$V_{11} = E_0 T_{tr} l \frac{2\pi\delta x}{a} \frac{2\pi\delta y}{b} \tag{18}$$

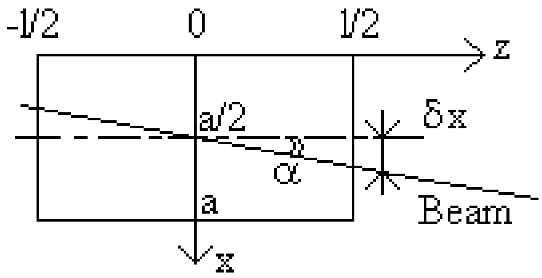
Consequently, using a reference cavity with the same dimensions that measures the first *common mode* voltage (V_{00}), and using the cavity BPM to measure V_{10} and V_{01}, we can get directly the beam offset by reading :

$$\frac{V_{10}}{V_{00}} = \frac{\pi\delta x}{a} \tag{19}$$

$$\frac{V_{01}}{V_{00}} = \frac{\pi\delta y}{b} \tag{20}$$

2.4 Beam inclination

We will now see the effects of beam inclination on the mode voltage. Assume that the beam has a little angle α with the z axis : x and z coordinates are now linked according to the following picture : $\delta x = z\tan(\alpha)$.



We are going to calculate the mode voltage using 13. First, we develop the sine around the position $x = a/2 + z\tan(\alpha) \approx a/2 + z\alpha; y = b/2$ what gives : $\sin(\frac{m\pi x}{a}) = \sin(\frac{m\pi}{2}) + \frac{m\pi\alpha z}{a}\cos(\frac{m\pi}{2})$ and the same discussion than before : either m is odd and we can't see the inclination's effect on the voltage, either m is even and the previous term is then proportional to the beam inclination. We will now continue our calculations for the TM_{10} mode.
The last approximation gives us :

$$V'_{10} = \int_{-l/2}^{l/2} E_0 \frac{2\pi\alpha}{a} zexp(ikz)dz \tag{21}$$

One method to calculate this is to write

$$\int_{-l/2}^{l/2} z\exp(ikz)dz = -i\frac{\partial}{\partial k}\int_{-l/2}^{l/2} \exp(ikz)dz \tag{22}$$

This leads us to the formula :

$$V'_{10} = \frac{2iE_0}{k^2}\frac{2\pi\alpha}{a}\Big(\sin(\frac{kl}{2}) - \frac{kl}{2}\cos(\frac{kl}{2})\Big) \tag{23}$$

It shows that de mode voltage is proportional to the beam inclination, what is interesting for beam monitoring.
Now, we can compare the two mode voltages V_{10} and V'_{10} to be independant on the amplitude and we also see that there is a phase of 90° between these two signals, that can be used for detection.

$$\frac{V'_{10}}{V_{10}} = \frac{i\alpha}{k\delta x}\Big(1 - \frac{kl}{2}\cot(\frac{kl}{2})\Big) \tag{24}$$

We have seen theoretically how the first TM modes are linked with the beam offset and inclination. Thus, the most interesting modes for rectangular cavity BPMs will be TM_{10} and TM_{01} and we will also need a reference cavity that will measure the *common mode* TM_{00}.
We will now see how to use these properties for experiments using cavity BPMs.

3 Experimental features

The experimental principle of cavity BPMs is quite simple : we assume that there is a bunch of charged particules going through a waveguide. When it encounters a rectangular cavity, some charges are left that produce an electric field which obeys to the equations that we have seen before. Then it is possible to measure the output voltage that is linked with the mode voltage and to calculate some properties of the bunch that passed. Particularly the beam offset and inclination are the aim of the measure here.

3.1 Second cavity parameters

Since the real cavity is not perfect, when a bunch of charged particules goes through the cavity, you can't see a dirac on the monitor but a curve with a peak and a certain width when you look at the ij^{th} mode frequency (that we will call n^{th} mode to simplify the notation). So we may see the cavity as an RLC circuit with two important parameters : the shunt impedance $R_{s,n}$ and the internal quality factor $Q_{n,0}$.
The definition of the shunt impedance for the n^{th} mode is :

$$R_{s,n} = \frac{V_n^2}{2P_{n,loss}} \tag{25}$$

where V_n is the voltage of the n^{th} mode and $P_{n,loss}$ is the power dissipated in the cavity.
The internal quality factor of the cavity for each mode is defined by the ratio between the power of the n^{th} mode and the power dissipated in the cavity. So, it is given by :

$$Q_{n,0} = \frac{\omega_n W_n}{P_{n,loss}} \tag{26}$$

where W_n is the energy lost in the cavity by the bunch. Note that $Q_{n,0}$ does not depend on V_n since it is a constant of the cavity. We have :

$$W_n = \frac{q}{2} V_n \tag{27}$$

where q is the bunch charge and the factor 1/2 is given by the *theorem of beam loading*. Then we will use the normalized impedance for it only depends on the geometry of the cavity :

$$\left(\frac{R_s}{Q_0}\right)_n = \frac{V_n^2}{\omega_n W_n} \tag{28}$$

For dipole modes, V_n is proportional to the beam offset δx or to the beam inclination α, so $\left(\frac{R_s}{Q_0}\right)$ is proportional to their square.
We can rearrange 28 with 27 to obtain :

$$V_n = \frac{q}{2}\omega_n \left(\frac{R_s}{Q_0}\right)_n \tag{29}$$

and

$$W_n = \frac{q^2}{4}\omega_n \left(\frac{R_s}{Q_0}\right)_n \tag{30}$$

Now we are going to link the energy lost in the cavity with the output power so that we will show that the measured output voltage is still linear with the beam offset (or inclination).

3.2 Output voltage

We want to calculate the output voltage that we measure with our setup. There are many possibilities to measure the signal created in the cavity and they all imply coupling the cavity with what we call an external circuit such as a waveguide and using an antenna to get the signal. Therefore, for this external circuit, there is also an impedance Z (usually, $Z = 50\Omega$) and a quality factor that will be the ratio between the mode power and the output signal power :

$$Q_{n,ext} = \frac{\omega_n W_n}{P_{n,out}} \tag{31}$$

with this definition and taking into account the fact that the cavity BPM is made of two components : one cavity and one external coupling circuit, we can define a global quality factor Q_L called "Q-loaded":

$$Q_{n,L} = \frac{\omega_n W_n}{P_{n,loss} + P_{n,out}} \tag{32}$$

which means that Q_L may also be defined by :

$$Q_{n,L} = \frac{Q_{n,0} Q_{n,ext}}{Q_{n,0} + Q_{n,ext}} \tag{33}$$

This factor is the one that we may measure when we measure reflexion and transmission in the cavity, but this point will be discussed in the next section. We may write the output voltage :

$$V_{n,out} = \sqrt{ZP_{n,out}} \tag{34}$$

and using $Q_{n,ext}$ definition and relation 30, we obtain the form :

$$V_{n,out} = \frac{q\omega_n}{2}\sqrt{\frac{Z}{Q_{n,ext}}\frac{R_s}{Q_0}_n} \tag{35}$$

where it is now clear that V_{out}, for dipole modes, is proportional to the offset (or inclination) because $\left(\frac{R_s}{Q_0}\right)$ is proportional to their square.
As a consequence, we have prooved that the output voltage had a linear dependance on the beam deviation. We will now see how to measure the different quality factors, so that all the parameters will be known and we will easily calculate the beam deviation.

3.3 About quality factors measurements

Nota Bene : the following relations will be valid for each mode so we will forget about the n index.
We have seen before that the measured quality factor was Q_L. We will see here how to calculate the other quality factors. Let us introduce, the factor β : the ratio between the output power and the power lost in the cavity :

$$\beta = \frac{P_{out}}{P_{loss}} \tag{36}$$

Using the definitions of Q_0 and Q_{ext}, we may rewrite β as :

$$\beta = \frac{Q_0}{Q_{ext}} \tag{37}$$

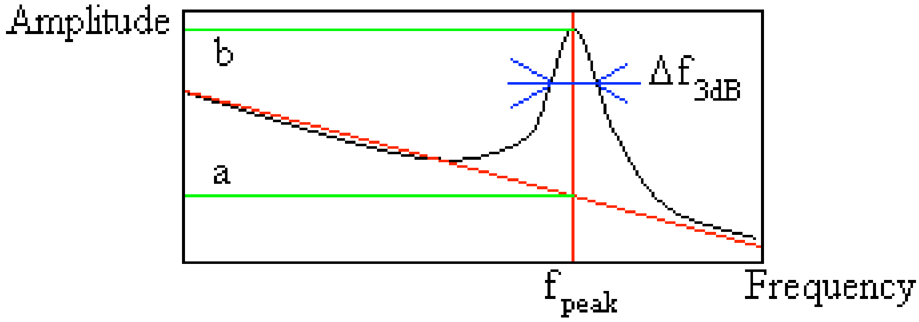
and now, we can also rewrite Q_L :

$$Q_L = \frac{Q_0}{1+\beta} \tag{38}$$

It is now possible, with a network analyzer to deduce the values of Q_L, β and then Q_0 and Q_{ext} for reflexion and transmission modes.
When we are studying either reflexion or transmission of the cavity, for different mode frequencies, we can see a peak with a certain width. We can deduce Q_L using :

$$Q_L = \frac{f_{peak}}{\Delta f_{-3dB}} \tag{39}$$

Warning : All the calculations are made in terms of power. So, using the notations of the next picture, we must measure the frequency width for an amplitude of $\sqrt{\frac{a^2+b^2}{2}}$ and not $\frac{a+b}{2}$!



Assume we are studying reflexion, let R be the reflection of the cavity. β should be calculated using :

$$\beta = \frac{1-R}{R} \tag{40}$$

because the output power is the transmitted signal and the dissipated power comes from reflexion.
Assume we are studying reflexion, let T be the reflection of the cavity. β should be calculated using :

$$\beta = \frac{T}{1-T} \tag{41}$$

for the same reasons that before.
Consequently, it is possible to measure all the geometry dependent parameters with a network analyzer. Moreover, it is important to understand that these parameters depend only on the geometry. This is, indeed, helpful for us to create cavities with known parameters because the more precision we have for them, the more precision we have on the beam's measurements.

4 Conclusion

In this text, we have seen that rectangular cavities can be used for beam position monitoring, in very simple cases using the first TM modes. Since all the parameters can be designed according to our needs, we can adapt the cavity to the beam needs and get a much higher precision for monitoring the beam.
This was just an introduction on how rectangular cavity BPMs work. Many things should be added such as tunning a cavity or measuring the coupling between the x and y ports. However, these first elements are essential to understand rectangular cavity BPM for they are the basis of all these kind of detectors.

References

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[4] Y.Honda, RF measurement of IP-BPM cold model, 2006/04/06