Master Thesis
Development of Beam-position Monitors with High Position Resolution

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February 3, 2008
## CONTENTS

**1 ILC, ATF and ATF2**

1.1 Overview of ILC Project .......................... 1
   1.1.1 Target Physics ................................. 1
   1.1.2 Requirements for ILC .......................... 2
   1.1.3 Accelerator Design ............................ 3

1.2 Overview of ATF2 Project ......................... 4

1.3 BPMs at ILC ...................................... 6

1.4 BPMs at ATF2 .................................... 6

**2 Principle of Cavity BPMs** .......................... 8

2.1 Limitation of electrode BPMs ....................... 8

2.2 Features of Cavity BPMs ........................... 8

2.3 Eignemodes of the cavity .......................... 8

2.4 Selection of di-pole modes ........................ 9

**3 Parameters of Cavity BPMs** ....................... 11

3.1 Quality Factor $Q$ and Coupling Constant $\beta$ .... 11

3.2 $R/Q$ ............................................. 12

**4 Calculation of output signals** ................... 13

4.1 Fundamental Theorem of beam excitation .......... 13

4.2 Effect of bunch length ........................... 14

4.3 Output signals .................................. 15

4.4 $R/Q$ of a Cylindrical Cavity ..................... 16
   4.4.1 Di-pole mode .................................. 16
   4.4.2 Mono-pole mode ................................. 18

4.5 $R/Q$ of a Rectangular Cavity ..................... 19
   4.5.1 Di-pole mode .................................. 20
   4.5.2 Mono-pole mode ................................. 21

4.6 Noise Components ................................ 21
   4.6.1 Beam Angle Signals ............................ 22
   4.6.2 Bunch Tilt Signals ............................. 23
   4.6.3 Common Mode Signals ........................... 23

4.7 Calculation of $Q_0$ ................................ 24
   4.7.1 Calculation of energy loss .................... 24

4.7.2 Cylindrical Cavity Case ....................... 25
   4.7.3 Rectangular Cavity Case ....................... 26

**5 IP-BPM Design** .................................. 28

5.1 Fabrication of IP-BPM ............................. 28
   5.1.1 Fabrication Method ............................. 28
   5.1.2 IP-BPM Block ................................. 29

5.2 Characteristics of IP-BPM ........................ 30

5.3 Design Parameters ................................ 30
   5.3.1 Resonant Frequency $f_0$ ...................... 30
   5.3.2 Cavity Length $L$ ............................... 31
   5.3.3 Coupling Constant $\beta$ ...................... 31

5.4 Reference Cavity ................................ 33
## 6 Basic Tests

6.1 Measurement of basic parameters  
6.1.1 Measurement Principle  
6.1.2 Prediction from theory  
6.1.3 HFSS Simulation  
6.1.4 Results  
6.2 Measurement of X-Y isolation  
6.3 Measurement of $R/Q$  
6.3.1 Estimation from theory  
6.3.2 Simulation by HFSS  
6.3.3 Measurement Principle  
6.3.4 Measurement Scheme  
6.3.5 Results  
6.4 Properties of Reference Cavity

## 7 Detecting Scheme

7.1 Phase Detection  
7.1.1 Analog Detection  
7.1.2 Digital Detection  
7.1.3 I-Q Tuning  
7.2 Spec of electronics  
7.2.1 Noise Figure  
7.2.2 Spec of converter module  
7.2.3 Spec of the total system

## 8 IP-BPM Installation

8.1 IP-BPM Cavity  
8.2 Electronics

## 9 Beam Tests

9.1 Monitors used at IP-BPM study  
9.1.1 Stripline BPMs  
9.1.2 Wire Scanners  
9.1.3 ICT Monitor  
9.2 Stripline BPM Calibration  
9.3 ICT calibration  
9.4 Position Sensitivity  
9.4.1 Theoretical Prediction  
9.4.2 Measurement Scheme  
9.4.3 Results  
9.4.4 Discussions  
9.5 Angle Sensitivity  
9.5.1 Prediction from theory  
9.5.2 Measurement Scheme  
9.5.3 Results

## 10 Position Resolution Measurement (Single Bunch)

10.1 Principle  
10.2 Reference Cavity Calibration  
10.3 I-Q tuning  
10.4 Calibration Run
10.5 Resolution Run .................................................. 76
10.6 Regression Parameters ........................................ 81
10.7 Thermal Noise .................................................... 84
10.8 Suspection of noise origin ..................................... 88
  10.8.1 Temperature Vibration ................................. 88
  10.8.2 Mechanical Vibration .................................. 89
  10.8.3 Other Suspects .......................................... 89
11 Position Resolution Measurement (Multiple Bunch) .......... 91
  11.1 Expected Resolution in Multiple Bunch Mode ............. 91
  11.2 Bunch Separation and Phase Detection ..................... 91
  11.3 Beam Synchronized Phase Origin ......................... 92
  11.4 Analysis Method ........................................... 94
  11.5 Measurement Procedure .................................. 96
  11.6 Reference Cavity Calibration .............................. 97
  11.7 Pedestal Run ................................................ 97
  11.8 Calibration Run with single bunch ......................... 97
  11.9 Calibration Run ........................................... 99
  11.10 Resolution Run ........................................... 99
12 Conclusion ......................................................... 108
13 Acknowledgements ................................................ 109
A APPENDIX ................................................................ 110
  A.1 Electromagnetic field of cavities .......................... 110
    A.1.1 Starting point □ Maxwell Equation .................... 110
    A.1.2 Rectangular Cavity .................................. 110
    A.1.3 Cylindrical Cavity .................................. 113
  A.2 Calculation of Energy Loss at Cavity Wall .............. 116
    A.2.1 Starting point □ Maxwell Equation .................... 116
  A.3 HFSS Field Calculator ....................................... 118
    A.3.1 Stored Energy U ...................................... 118
    A.3.2 Field Integration along axis .......................... 118
  A.4 Slater’s Perturbation Theory .............................. 118
    A.4.1 Vector Formulas ...................................... 118
    A.4.2 Orthogonal function expansion of electro-magnetic field of a resonant cavity 119
    A.4.3 Maxwell Equation in a resonant cavity ................ 122
    A.4.4 Bead Perturbation in a Resonant Cavity ............... 125
  A.5 Friss’s Formula ............................................. 127
  A.6 Standard Regression Coefficients ......................... 128
  A.7 IP-BPM Design Drawing ..................................... 128

List of Figures

1.1 ILC schematics for Phase I (500 GeV), from [1] ............ 4
1.2 ATF2 Layout, from [2] ........................................ 5
2.1 Structure of a cavity BPM, from [5] ........................ 9
2.2 Eigen mode of a cylindrical cavity ............................ 9
2.3 Eigen mode of a rectangular cavity ............................ 10
LIST OF FIGURES

2.4 Coupling of cavity, wave guide and coaxial antenna ........................................... 10
2.1 Output signal from a cavity ................................................................................. 16
4.2 Modeling of beam angle signal ........................................................................... 22
4.3 Modeling of tilted bunch signal .......................................................................... 23
4.4 Tail of common mode signal ............................................................................... 24
5.1 IP-BPM Cavity Design ......................................................................................... 28
5.2 Principle of IP-BPM measurement ....................................................................... 29
5.3 Required IP-BPM setup for ATF2 ........................................................................ 29
5.4 IP-BPM operation at ATF2 .................................................................................... 29
5.5 Excited di-pole mode power vs cavity geometry ................................................. 30
5.6 Wave guide design parameters ............................................................................ 32
5.7 Reference cavity structure, from [5] ..................................................................... 33
6.1 IP-BPM block components .................................................................................... 34
6.2 IP-BPM block appearance .................................................................................... 34
6.3 1 Port Measurement .............................................................................................. 35
6.4 2 Port Measurement .............................................................................................. 35
6.5 S-parameters of transmission and reflection ........................................................ 35
6.6 Simulation geometry for Q measurement ............................................................ 38
6.7 S parameter for X-port Transmission calculated by HFSS .................................. 39
6.8 S parameter for Y-port Transmission calculated by HFSS .................................. 39
6.9 Schematics of X-Y isolation measurement ........................................................... 40
6.10 Simulation geometry for R/Q measurement ....................................................... 41
6.11 E_z distribution in longitudinal direction, from [12] ................................................ 42
6.12 R/Q Simulation by HFSS .................................................................................... 42
6.13 Schematics of Bead Perturbation Measurement ................................................ 44
6.14 X di-pole mode frequency shift .......................................................................... 44
6.15 Y di-pole mode frequency shift .......................................................................... 44
6.16 Measured R/Q of X di-pole mode ...................................................................... 45
6.17 Measured R/Q of Y di-pole mode ...................................................................... 45
6.18 Schematics of reference cavity frequency tuning ................................................ 45
7.1 Phase difference of sensor cavity signals .............................................................. 46
7.2 Principle of phase detection using a reference cavity ........................................... 47
7.3 Block Diagram of Analog Detection .................................................................. 49
7.4 Variable Attenuator ............................................................................................. 50
7.5 Block Diagram of Down Converter ..................................................................... 50
7.6 Block Diagram of Limiter Detector ..................................................................... 51
7.7 Block Diagram of Phase Detector ........................................................................ 52
7.8 Block Diagram of Digital Detection .................................................................... 53
7.9 Amplitude and phase detection by analysis ......................................................... 54
7.10 I-Q tuning using a phase shifter at analog detection ........................................... 54
7.11 Gain curve of the first amplifier .......................................................................... 56
7.12 Equivalent noise measurement of the first amplifier, from [15] ......................... 56
7.13 Schematics of the noise figure measurement ...................................................... 56
7.14 Equivalent noise measurement of the total system (X), from [15] ..................... 57
7.15 Equivalent noise measurement of the total system (Y), from [15] ..................... 57
8.1 Layout of ATF extraction line near by IP-BPM, from [5] ........................................ 58
8.2 Monitored temperature variation ......................................................................... 59
8.3 Monitored output power of electronics ............................................................... 59
9.1 Schematics of stripline BPM ................................................................................. 60
9.2 Schematics of wire scanner ................................................................................... 61
9.3 Schematics of ICT monitor ............................................................... 61
9.4 Stripline calibration setup ............................................................... 62
9.5 Wire scanner response against ZV7X current ...................................... 63
9.6 Wire scanner response against ZV8X current ...................................... 63
9.7 Wire scanner response against ZV9X current ...................................... 63
9.8 Wire scanner response against distance from magnet .............................. 63
9.9 Stripline BPM response against ZV7X current ..................................... 64
9.10 Stripline BPM response against ZV8X current .................................... 64
9.11 Stripline BPM response against ZV9X current .................................... 64
9.12 Correction factor of Stripline BPMs .................................................. 64
9.13 ICT calibration setup ...................................................................... 66
9.14 Measured results of ICT calibration .................................................. 66
9.15 Confirmation of ICT calibration method .............................................. 66
9.16 Schematics of position sensitivity measurement .................................... 67
9.17 Output signal voltage vs beam position .............................................. 68
9.18 Effect of finite rise time at the diode detector ...................................... 68
9.19 Schematics of angle sensitivity measurement ....................................... 69
9.20 Definition of equivalent position signal .............................................. 69
9.21 Horizontal response with angle signal .............................................. 70
9.22 Vertical response with angle signal .................................................... 70
9.23 Angle sensitivity of IP-BPM ............................................................ 71
10.1 ICT vs Reference Signal ................................................................. 73
10.2 Method of cavity alignment .............................................................. 74
10.3 Calibration run setup .......................................................... 74
10.4 BPM1 Calibration Run under 30dB attenuation condition ................. 75
10.5 BPM2 Calibration Run under 30dB attenuation condition ................. 75
10.6 BPM3 Calibration Run under 30dB attenuation condition ................. 75
10.7 Extrapolation of calibration slope to 0 and 10 dB ................................ 75
10.8 Results of 1 hour resolution run under no attenuation ......................... 77
10.9 RMS of residual measured at each set of resolution run ..................... 79
10.11 Measured position resolution under various attenuations .............. 80
10.12 Beam position jitter at BPM2 under 20 dB attenuation .................... 81
10.13 Beam angle jitter at BPM2 under 20 dB attenuation ....................... 81
10.14 Misalignment between 2 cavities .................................................. 82
10.15 Correlation between noise components ......................................... 83
10.16 Schematics of thermal noise measurement ...................................... 85
10.17 Calibration Run under 30dB attenuation ......................................... 86
10.18 Measured rms of residual at short Resolution Runs ......................... 86
10.19 Results of long Resolution Run under no attenuation ...................... 86
10.20 Correlation between residual and regression parameters left after analysis 87
10.21 Measured thermal noise under various attenuations ....................... 88
10.22 Temperature vibration during 1 hour resolution run ......................... 88
10.23 Schematics of the mechanical vibration measurement ..................... 89
10.24 Measured mechanical vibration of sensor cavities ............................. 89
11.1 Multiple bunch signals ................................................................. 92
11.2 Schematics of DR 714 MHz stability measurement ............................ 93
11.3 Detected phase difference between DR 714 MHz and beam timing ........ 93
11.4 Schematics of multiple bunch measurement ...................................... 95
11.5 Contamination of former bunch in I-Q surface .................................. 95
List of Tables

1.1 Parameters of ILC, FFTB, and ATF2 .................................................. 5
1.2 BPMs required at ILC, from [4] ......................................................... 7
5.1 Design parameters of IP-BPM .......................................................... 28
5.2 Design parameters of slot and antenna .............................................. 32
6.1 Calculated parameters of IP-BPM ....................................................... 38
6.2 Simulated parameters of IP-BPM ....................................................... 39
6.3 Measured basic parameters of IP-BPM ............................................... 40
6.4 Results of HFSS simulation check using simple geometry .................. 42
6.5 Measured basic parameters of reference cavities .............................. 45
7.1 NBL00419 spec sheet, from [14] ....................................................... 55
7.2 Noise Figure of NBL00419 ............................................................... 56
9.1 Components near by IP-BPM at ATF extraction line .......................... 62
9.2 Measured slopes between beam displacement and steering magnet current ................................................. 63
9.3 Measured and expected slope of stripline BPMs against steering magnets ................................................. 65
9.4 Correction factors of stripline BPMs in comparison with wire scanners --------------------------------------------- 65
9.5 Results of position sensitivity measurement ..................................... 67
10.1 Measured calibration slope of each cavities ..................................... 76
10.2 Results of position resolution measurement under no attenuation ........... 80
10.3 Regression coefficients of no attenuation case ................................. 81
10.4 Coefficients calibrated to actual beam displacement ......................... 82
10.5 Coefficients of applied regression analysis ....................................... 83
10.6 Results of thermal noise measurement under no attenuation ............... 87
11.1 Decay rate and phase transition during bunch interval ...................... 98
11.2 Measured calibration slope at each cavities ........................................... 101
11.3 Measured position resolution under 3 bunch operation ............................. 107
Abstract

We have developed an ultra high position resolution cavity BPM for the final focus system of ATF2. ATF2 is a test beam line for ILC final focus system starting its operation in 2008, and is going to demonstrate 37 nm beam size and nano-meter beam orbit stability at virtual IP. For these purposes, a few nano-meter position resolution is required for this special cavity BPM, which is called the IP-BPM.

ATF2 is an extension of ATF, the Accelerator Test Facility, located at KEK, Tsukuba, Japan. ATF has achieved ultra low beam emittance required for ILC, and development of various beam diagnosis devices is taking place at existing ATF beam line.

We have fabricated 2 blocks of IP-BPM, consisting of two cavities in each block. IP-BPM can measure beam position in vertical and horizontal independently with a single cavity. The IP-BPM blocks are installed at ATF extraction line, and its position resolution was measured. As a result, position resolution of IP-BPM was proved to be $8.72 \pm 0.28 \pm 0.35$ nm, in case of single bunch operation. This is the best record ever achieved in the world today.

Also under multiple operation mode, IP-BPM could measure beam position of each bunch with sub-micron resolution, and is expected to achieve nano-meter resolution in ideal condition.
1 ILC, ATF and ATF2

1.1 Overview of ILC Project

Brand new physics of TeV scale is promised to be observed at LHC, the Large Hadron Collider, which is starting its operation in 2008 at CREN. The new physics includes the last missing piece of the Standard Model, Higgs particle, and SUSY particles or extra dimensions. However, LHC is still not versatile. Since LHC is a proton-proton collider, events are so complicated, which limits the information we can acquire. Also the center of mass energies of collisions for fundamental processes are varied in a wide range, and the background rate is very high. Therefore, to measure precise physics properties of the new physics, an electron-positron collider is absolutely necessary. ILC is the next generation electron-positron collider, which has the spec to determine properties of new physics observed, or even which is not observed at LHC.

ILC generates genuine elementary particle interactions at the collision point, which enables us to precisely determine the properties of brand new phenomena, or Standard Model physics. Its ultra high sensitivity to new particles or events can reveal the hidden physics law and determine which model is correct. ILC and LHC are complementary with each other, and therefore realization of ILC is the most important challenge for high energy physics. ILC is planned to start its operation at 2020, and various R&D for accelerators and physics experiments are now in progress all over the world.

The target physics, and accelerator design to achieve physics requirements of ILC are summarized in this section. More details can be referred in [1].

1.1.1 Target Physics

Phase I and Phase II are planned for ILC, and the target physics differs between Phase I and Phase II. At Phase I, maximum center of mass energy will be 500 GeV. The collision energy is controllable, and will be focused on weak scale. At Phase II, maximum center of mass energy will be 1 TeV, and will be focused on new physics beyond Standard Model.

The main target of Phase I is to reveal the origin of mass, or determination of the mechanism of spontaneous symmetry breaking of electro-weak interaction. If Higgs does exist, ILC can certainly be a "Higgs Factory". By controlling collision energy, momentum and polarization, its properties are determined precisely at ILC. Also observation of coupling constant between Higgs and W, Z, top, or self-coupling constant would enable us to reveal the mechanism and structure of the "vacuum". Precise measurement of Higgs properties also enables us to understand if there is new physics beyond Standard Model existing.

Also, ILC is going to be a "Top Factory", to determine the top mass in precision of 100 MeV, and its coupling constant precisely. This is extremely important to understand electro-weak interaction and to verify Standard Model precisely. If difference was observed, it implies some new physics existing at TeV scale.

There are various physics objectives for ILC Phase II.

Of course Higgs search is continued at Phase II. Even in case of heavy Higgs which could not be created at Phase I, precisely measured electro-weak data implies Higgs to be found under 1 TeV. However, there is still a possibility that there are no Higgs particles existing in the nature. Even in this case, new physics and new particles are considered to be existing at 1 ~ 10 TeV region, and ILC can directly or indirectly observe the new physics.

Hierarchy problem is also one of the most important issues to be solved at ILC. Applying Standard Model, Higgs mass would easily be at the GUT Scale due to quantum corrections, without incredibly fine tuning. There are several models proposed to solve this problem, and ILC can determine which model is correct. The first model is the SUSY (SUper SYmmetry) Model, and the lightest SUSY particle is expected to exist at 500 GeV ~ 1 TeV range, which ILC has enough spec to create. By changing the collision energy, precise measurement of SUSY particle properties is possible. Creation and precise measurement of SUSY particles enable us to understand the mechanism of SUSY breaking at low energy, and also the nature of grand unification. The second model is the Extra Dimension Model, which ILC has the ability to observe Kaluza-Klein particles or its quantum effects. Kaluza-Klein particle is an excitation state of graviton or Standard Model particles oscillating in extra dimensions. The third model is the Little Higgs Model, which predicts...
new particles at 10 TeV region. Although ILC cannot directly create these new particles, new physics is observed through the coupling of Higgs to these new particles.

Dark matter is also expected to be created at ILC. As shown from the results of COBE and WMAP, normal matter, which is consist of Standard Model particles, account for only 4% of the energy density of universe. The rest is called dark matter and dark energy. So far, neutralino, the super-symmetric partner of neutrino, is a well motivated candidate of the dark matter. From our understanding of mass density of the universe, its mass is expected to be order of 100 GeV, which is possible to be created at ILC. In this case, its properties can also be determined precisely at ILC.

1.1.2 Requirements for ILC

After LEP (Large Electron Positron Collider), of which center of mass energy was approximately 200 GeV, an energy frontier $e^+e^-$ collider has not been constructed. Considering synchrotron radiation, energy loss of the beam per one turn $\Delta E$ follows

$$\Delta E \propto \frac{E^4}{R^4 m^4},$$

(1.1)

where $E$ is the beam energy, $R$ is the bending radius and $m$ is the mass of the particle. There are two solutions in order to upgrade beam energy $E$, which are using heavy particles with large $m$, and applying large $R$. However in case of $e^+, e^-$ colliders, large $R$ is the only solution. Furthermore, since energy loss increases proportional to $E^4$, drastic increase of $R$ is required, which is not at all practical. To be free of this problem, the next generation energy frontier accelerator, ILC, is going to be a linear collider, which is $R = \infty$. However, there are so many technical difficulties to develop a high energy linear collider compared to circular accelerators.

First, accelerating gradient must be improved up to 31.5 MV/m and 36 MV/m as design values of ILC Phase I and Phase II, respectively. They are beyond current technology. In case of circular accelerators, charged particles could pass the accelerating unit for many times, while turning around the orbit. However for linear accelerators, charged particles pass the accelerating unit only once. In order to suppress the accelerator size, accelerating gradient of an accelerating unit must be drastically increased. Assuming design values of ILC, its total length is still about 40 km for Phase II.

Second, luminosity $L$ must be improved up to $2 \times 10^{34}$ cm$^{-2}$s$^{-1}$ as ILC design value. In case of circular accelerators, charged particles could run through collision in micro-second order. However for linear colliders, this is impossible. Rate of a physical event of cross section $\sigma$ is represented as $L\sigma$. Considering a collision of 2 bunches consisting of $N$ particles in each bunch, luminosity $L$ is represented as

$$L = f_{col} \frac{N^2}{S},$$

(1.2)

where $f_{col}$ is the collision frequency, and $S$ is the cross section of the beam. In order to increase rate of a physical event, beams at the collision point (IP, Interaction Point) must be strongly focused compared to circular accelerators. The nominal design value of beam size at ILC IP is 5.7 nm and 655 nm in vertical and horizontal, respectively. Charged particles are focused to small beam size using quadrupole magnets. However, beam bunch of a finite length can be focused only in a limited region. This is called the "Hourglass Effect".

To achieve ultra small beam size, achievement of ultra low beam emittance is essential. Emittance is defined as beam dimension in phase space of beam position and momentum, and the unit is m·eV/c. Also, normalized emittance is defined by multiplying the emittance by $\gamma$. This normalized beam emittance is conserved during the focusing or acceleration. Therefore by strongly focusing beam size, momentum of the particles consisting the beam would increase dramatically and limit the beam size. Thus development of ultra low emittance beam is inevitabe. The design value of normalized emittance for ILC is $\epsilon_x = 1 \times 10^{-5}$ (m) and $\epsilon_y = 4 \times 10^{-8}$ (m) in horizontal and vertical, respectively.

Today, both improvements are about to be demonstrated at KEK, located at Tsukuba, Japan. Superconductive RF accelerating cavities of ultra high accelerating gradient is being developed
at STF, the Superconductive Test Facility. Demonstration of nano-meter electron beam size is going to be demonstrated at ATF2, which is an extension of ATF, the Accelerator Test Facility. In advance, design value of ILC beam emittance is already achieved at the ATF Damping Ring. Therefore, contribution of Japan for developing accelerating technology for ILC is extremely significant.

1.1.3 Accelerator Design

The design of ILC accelerator is shown in Figure 1.1. Main components are described in the following.

Bean Injection An electron gun is installed to generate electron beam. The electron beam injected is immediately accelerated by 5GeV Linac. Possibility of beam polarization over 80 % is required from physics studies, which is a technological challenge for electron guns. The only polarized electron source in practical use today is NEA GaAs cathode. Applying this cathode, its spec must meet the parameters of ILC, which is 3.2 nC beam charge, 4.3 ps bunch length, 308 ns bunch interval, 2820 bunch, and so on.

DR (Damping Ring) After accelerated to 5 GeV, electron and positron beam are stored in DR, to achieve ultra low normalized emittance. There are 3 DRs, all 6.6 km around. One is for electron beam and two are for positron beam. As discussed in advance, beam emittance is conserved during the focusing or acceleration, in case of conservative force. However with nonconservative force, such as synchrotron radiation, it is not conserved. As shown in Figure 1.1, energy loss per turn is proportional to $E^4$. Therefore through synchrotron radiation, energy dispersion is reduced. Furthermore by losing momentum through synchrotron radiation, beam emittance also reduces. In this situation momentum in beam direction is also lost, but is recovered by the acceleration at straight section.

RTML (Ring To Main Linac) After the DR, beam is transported in opposite direction of the collision point, turn around 180 degrees and then enter the main linac. This part is called the RTML. At RTML, spin of electron and positron are controlled to meet the requirements from physics experiments, and bunch length is compressed from a few mm to some 100 µm, in order to decrease Hourglass Effect.

Main Linac $e^+e^-$ beam is accelerated to 500 GeV or 1 TeV in center of mass energy at Main Linac. At Phase I (500 GeV), the length would be 11 km for both $e^+$ and $e^-$. 

Undulator Positron beam is generated using a helical undulator. An undulator is a group of magnets, which wind the $e^-$ orbit by changing the direction of induced magnetic field. When accelerated $e^-$ beam of approximately 150 GeV enters the undulator, $\gamma$ rays of some 10 MeV is radiated by synchrotron radiation. From the collision of this $\gamma$ ray and positron target, $e^+$ is emitted. Applying helical undulator, polarized $e^+$ is generated, but much more technical improvement is needed. As an alternative, laser compton is also considered as positron source.

BDS (Beam Delivery System) From the main linac, $e^+e^-$ beam is transported to the IP by BDS. Its main functions are cutting out beam halo using collimators, focusing beam size (FFS: Final Focus System), controlling collision angle, and treating the beam after the collision with a beam dump. The most important is the FFS, which is mainly consisting of optics to achieve ultra small beam size at IP, and the total length is 5.5 km. Quadrupoles are aligned to focus beam size, while their chromatic aberration is corrected by sextupole magnets. Also to improve luminosity, crab crossing is going to be applied, which is already demonstrated at KEKB.

IP (Interaction Point) IP is the collision point of $e^+$ and $e^-$, which is the "New Physics Generator". Detectors for physics experiments are going to be installed around IP. The crossing angle of $e^+$ and $e^-$ is going to be 14 mrad.
1.2 Overview of ATF2 Project

As discussed in the former section, achievement of ultra low emittance beam and developing nanometer beam size is essential for ILC. On this purpose, final focus system of a linear collider was studied at FFTB (Final Focus Test Beam), located at SLAC. It was operated from 1994 to 1997 and achieved 70 nm beam size in vertical, which is still the smallest beam size ever achieved in the world today. However, compared with its designed beam size, 47 nm, achieved beam size was much larger. This difference was considered to be originated from beam jitter. Unfortunately since the beam size monitor installed at FFTB could not measure beam size with a single pulse, we could not tell if it really originated from beam jitter.

After FFTB experiment was completed, various linear collider projects proposed in several nations were integrated to ILC in 2004, relying on super-conductive technology. Therefore, a new test facility for demonstrating ILC final focus system was desired. In this purpose, ATF2[2] will start its operation in 2008 to demonstrate Local Chromaticity Correction, which is a new concept of chromaticity correction proposed after FFTB experiment, to achieve nano-meter beam size.
ATF2 is an extension of ATF (Accelerator Test Facility), located at KEK, Japan. As mentioned earlier, ATF has already achieved ultra low emittance beam required for ILC. Using this high quality electron beam, various tests for ILC final focus, such as development of nano-meter beam size and providing its sub-nanometer stability, will take place at ATF2. The whole image of ATF and ATF2 is shown in Figure 1.2.

There are two main goals for ATF2, which are

1. Achievement of 37 nm beam size (Phase I)
2. Control of beam position with nano-meter precision (Phase II).

At Phase I, final focus system based on Local Chromaticity Correction will be demonstrated, and 37 nm beam size will be achieved. The design parameters of ILC, FFTB, and ATF2 are compared in Table 1.1. As shown in the table, beam emittance required for ATF2 is the same or better than ILC. Due to this ultra low emittance, beam size at ATF2 is expected to be similar with FFTB, even though its beam energy is much lower.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>ILC</th>
<th>FFTB</th>
<th>ATF2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam Energy $E$ (GeV)</td>
<td>250 / 500</td>
<td>46</td>
<td>1.3</td>
</tr>
<tr>
<td># of $e^-$ per bunch $N$</td>
<td>$2 \times 10^{10}$</td>
<td>$5 \times 10^{10}$</td>
<td>$1 \times 10^{10}$</td>
</tr>
<tr>
<td>Bunch Interval (ns)</td>
<td>307.7</td>
<td>307.7</td>
<td>60</td>
</tr>
<tr>
<td>Bunch Number</td>
<td>2820</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Normalized Emittance $\epsilon_x$ (m)</td>
<td>$1 \times 10^{-5}$</td>
<td>$3 \times 10^{-5}$</td>
<td>$3 \times 10^{-6}$</td>
</tr>
<tr>
<td>Normalized Emittance $\epsilon_y$ (m)</td>
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<td>$2 \times 10^{-6}$</td>
<td>$3 \times 10^{-8}$</td>
</tr>
<tr>
<td>Beam Size $\sigma_x$ ($\mu$m)</td>
<td>0.66 / 0.55</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Beam Size $\sigma_y$ (nm)</td>
<td>5.7 / 3.5</td>
<td>47</td>
<td>37</td>
</tr>
</tbody>
</table>

Table 1.1: Parameters of ILC, FFTB, and ATF2

At Phase I, a nano-meter BSM (Beam Size Monitor), which is called the Shintake Monitor[^3], will be installed at ATF2 IP to confirm 37 nm beam size. Shintake Monitor itself cannot measure
beam size with a single pulse, which is the same case as FFTB. Therefore in order to monitor beam jitter during beam size measurement, a nano-meter BPM (Beam Position Monitor) will also be installed. Applying this BPM and feed-back system, it will be possible to measure beam size free from beam jitter. This BPM is called the IP-BPM, which is actually the key module for Phase II.

At Phase II, beam orbit stabilization with nano-meter precision will be demonstrated at IP. Also, beam jitter controlling techniques at nano-meter level with an ILC-like beam are going to be established. At ILC, beam position at IP would be measured using the facing beam. However, ATF2 is not a collider and uses electron beam only. In this case, electron beam must be stabilized at nano-meter precision to a benchmark at IP, which is the IP-BPM.

IP-BPM therefore must have nano-meter position resolution to confirm beam orbit stabilization. To achieve this ultra high resolution, special design is required for IP-BPM. IP-BPM is the main theme of this paper, but first overview of BPMs required for ILC and ATF2 are summarized in the following sections.

1.3 BPMs at ILC

At ILC, various types of BPMs (Beam Position Monitors) are required. They are summarized in Table 1.2.

At injection points of e\(^+\) and e\(^-\), stripline BPMs are going to be used. This is because high position resolution is not required at injection points, due to large beam size.

At the DR (Damping Ring), 4 BPMs per betatron oscillation cycle length are required. Most of them are used for equilibrium orbit adjustment. These BPMs need sub-micron position resolution, and it can be achieved by integrating the signal of button type BPMs. Some BPMs are used for feedback or measuring beam oscillation. These BPMs need to distinguish bunch to bunch of 6\(\text{ns}\) intervals and therefore high speed processing is required.

At the main linac, a BPM is aligned at each quadrupole magnet. Position resolution better than 500 nm is required for each bunch of 300 \(\text{ns}\) intervals. In this case, cavity BPMs are used.

At the BDS (Beam Delivery System), cavity BPMs are used to meet the requirements of high position resolution and stability.

1.4 BPMs at ATF2

The requirements for BPMs at ATF2 are basically the same with those of ILC BDS. Cavity BPMs are mainly used, and they are fixed at each quadrupole and sextupole magnets, which are called the Q-BPMs. These magnets are aligned on precise movers, and the BPMs move with these magnets. If the beam orbit runs off the magnet center, beam would be unnecessarily kicked and thus beam size would not be focused enough. Using cavity BPMs for beam orbit feedbacks, beam orbit is controlled to pass the magnetic center. During beam controlling, a few micron stability is required not only for mechanical precision, but also for detecting circuits.

As discussed, facing beam is used for the beam position/size measurement at ILC IP. However it is not the case for ATF2 IP, since there is only electron beam. Instead, an ultra high resolution cavity BPM is going to be installed at ATF2 IP, and electron beam is going to be stabilized against this BPM. This special BPM is called the IP-BPM, and nano-meter position resolution is required for this purpose.

In this paper, design, fabrication, and measurement results of important properties of IP-BPM are discussed in detail. At first in section 2 principle of beam position monitoring using cavity BPMs is discussed, since IP-BPM is a special type of cavity BPM. Next in section 3 parameters used to characterize cavity BPM properties are defined. In section 4 output beam position signal of cavity BPMs are obtained. It is shown that di-pole mode of a cavity is sensitive to beam position, and important parameters which determine position sensitivity of a cavity are \(R/Q\) and \(Q_{\text{ext}}\). Also, contributions of noise components other than beam position, such as beam angle and bunch tilt, are discussed. Through this discussion, it is shown that signal phase detection is essential for beam position measurement. In section 5 IP-BPM design is determined to meet ATF2 requirements, such as ultra high position resolution, on the basis of former discussions. In section 6 fabricated IP-BPM design is checked by measuring the basic parameters of IP-BPM,
and was proved to meet the ATF2 requirements. In section 7, details of detecting scheme of beam position signal are explained, including phase detection methods. This scheme is used in various beam tests. Also, our electronics was proved to have sufficient spec to detect nano-meter position signal. In section 8, IP-BPM is installed in the ATF extraction line for various beam tests. In section 9, characteristics of IP-BPM are checked using ATF beam line. Position sensitivity and angle sensitivity of IP-BPM were proved to meet ATF2 requirements. In section 10 and section 11, position resolution of IP-BPM is measured in case of single bunch operation and multiple bunch operation. It is shown that IP-BPM has nano-meter position resolution, which is the best record ever achieved in the world today. Finally in section 13, IP-BPM measurement results are concluded and required improvements are discussed.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of pcs</td>
<td>~600</td>
</tr>
<tr>
<td>Environment</td>
<td>normal or low temp</td>
</tr>
<tr>
<td>Pipe aperture</td>
<td>40 - 100 mm</td>
</tr>
<tr>
<td>Resolution</td>
<td>&lt;100 ( \mu )m</td>
</tr>
<tr>
<td>Stability</td>
<td>&lt;100 ( \mu )m</td>
</tr>
<tr>
<td>Time resolution</td>
<td>bunch to bunch</td>
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</table>

**Damping Ring**

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</thead>
<tbody>
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<tr>
<td>Environment</td>
<td>normal temp</td>
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<tr>
<td>Pipe aperture</td>
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<tr>
<td>Resolution</td>
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</tr>
<tr>
<td>Stability</td>
<td>&lt;100 ( \mu )m</td>
</tr>
<tr>
<td>Time resolution</td>
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**Main Linac**

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<td>Pipe aperture</td>
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<td>Resolution</td>
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</tr>
<tr>
<td>Stability</td>
<td>&lt;10 ( \mu )m</td>
</tr>
<tr>
<td>Time resolution</td>
<td>bunch to bunch</td>
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</tbody>
</table>

**Beam Delivery System**

<table>
<thead>
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<th>Parameter</th>
<th>Quantity</th>
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</thead>
<tbody>
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<td>No. of pcs</td>
<td>~400</td>
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<tr>
<td>Environment</td>
<td>normal temp</td>
</tr>
<tr>
<td>Pipe aperture</td>
<td>on demands</td>
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<tr>
<td>Resolution</td>
<td>&lt;0.25 ( \mu )m</td>
</tr>
<tr>
<td>Stability</td>
<td>&lt;10 ( \mu )m (&lt;1 ( \mu )m)</td>
</tr>
<tr>
<td>Time resolution</td>
<td>bunch to bunch</td>
</tr>
</tbody>
</table>

Table 1.2: BPMs required at ILC, from 4
2 Principle of Cavity BPMs

As mentioned in section 1, IP-BPM is a special type of cavity BPM. In this section, advantages and principles of cavity BPMs are described. Cavity BPMs use di-pole modes of electro-magnetic wave excitations for beam position monitoring, and therefore details of di-pole modes and selecting method is discussed in detail.

2.1 Limitation of electrode BPMs

Position resolution of electrode BPMs such as button type BPMs or stripline BPMs is limited at approximately 1 \( \mu m \), which does not meet ATF2 requirements. This is basically due to the subtraction between 2 independent large values, which are the voltage excited at a pair of electrodes aligned across to each other. Beam position at an electrode BPM is determined from

\[
S \times \frac{V_1 - V_3}{V_1 + V_3},
\]

where \( V_1 \) and \( V_3 \) are the voltage excited at facing pair of electrodes, and \( S \) is a scale factor. \( S \) is determined by the shape of BPM, and is typically half value of beam pipe diameter. Since the voltages of electrodes are read out independently, resolution is limited by ADC bit count.

To improve resolution, it is effective to directly subtract analogue signals, before digitalizing at ADC. Moreover, signals from the cavity BPMs are automatically subtracted at the cavity. Di-pole mode signals correspond to the difference of the signals, and it is zero when electron beam passes the cavity center. Principally, it is possible to improve resolution to thermal noise level by using high gain electronics for detection.

2.2 Features of Cavity BPMs

Compared with electrode BPMs, cavity BPMs have two main advantages:

- Good stability is expected, since precision of electrical center and its stability depend essentially on the mechanical precision and not on following detecting circuit.
- Resolution of thermal noise level can be achieved by narrowing dynamic range and using high gain electronics, since signal is very small near the cavity center.

Due to these features, cavity BPMs are principally able to achieve nano-meter position resolution.

The cavity mainly consists of a cavity and waveguides. When electron beam passes the cavity, eigenmodes of various resonant RFs (Radio Frequencies) are excited. By selecting modes sensitive to beam position such as di-pole modes, beam position at the cavity can be monitored. The excited di-pole mode RF signal is coupled to the waveguides through slots, and the signal is read out through coaxial antennas positioned at the end of waveguides. The structure of a cavity BPM is shown in Figure 2.1.

2.3 Eigenmodes of the cavity

Eigenmodes of a cavity are determined by its boundary condition. Usually cavities, such as Q-BPMs for ATF2, are designed to be cylindrical. However, IP-BPM is designed to be rectangular. The details of the geometrical structures are discussed in section 5. For cylindrical or rectangular cavities, it is possible to calculate eigenmodes analytically, as shown in Appendix A.1.

Cylindrical coordinates are used to evaluate eigenmodes of cylindrical cavities. Electric fields and magnetic fields excited at the cavity are represented as \( \mathbf{E} = (E_r, E_\phi, E_z) \) and \( \mathbf{B} = (B_r, B_\phi, B_z) \), respectively. For beam position monitoring, TM mode \( \mathbf{B} = (B_r, B_\phi, 0) \) is essential. They are characterized by three parameters, \( m, n, l \), which are node numbers in \( r, \phi, \) and \( z \) direction respectively. Mode \( \text{TM}_{010} \) is the monopole mode, which is usually used for accelerating charged particles at accelerating cavities. It is also used at reference cavities for IP-BPM, which is discussed in section 5. \( \text{TM}_{110} \) is the di-pole mode of our concern. Di-pole mode is used for position detection.
When beam passes the cavity with an offset, amplitude of the excited signal is proportional to the beam offset. When beam passes the cavity center, signal is not excited.

This is basically the same with rectangular cavities also. Usually orthogonal coordinate system is used, and the fields are represented as \( \mathbf{E} = (E_x, E_y, E_z) \) and \( \mathbf{B} = (B_x, B_y, B_z) \). They are characterized by three parameters, \( m, n, l \), which are node numbers in \( x \), \( y \), and \( z \) direction respectively. Just like cylindrical cavities, TM mode \( B_z = 0 \) is essential, and di-pole modes are used for beam position monitoring. If cavity length differs in \( x \) and \( y \) direction, there are two different di-pole mode frequencies corresponding to \( x \) and \( y \), which are \( TM_{210} \) and \( TM_{120} \), respectively.

The eigenmodes of those cavities are shown in Figure 2.2 and Figure 2.3. In these figures, magnetic fields are drawn in blue and electric fields are drawn in red.

2.4 Selection of di-pole modes

To measure beam position, it is critical to read out only the di-pole mode of concern, and not other modes. Common mode (mono-pole mode) and other modes are unwanted noise components, and separation of di-pole mode signals from noise components is very important. Also, to measure beam position in \( X \) direction and \( Y \) direction independently, selection of \( X \) and \( Y \) signals is also needed. To meet these requirements, we used slot type couplers, and geometrically selected the desired di-pole mode. Only desired di-pole mode can couple magnetically to the waveguides.

To be transmitted to waveguides from the cavity, RF field have to pass a narrow rectangular
2 PRINCIPLE OF CAVITY BPMS

It is possible only when the magnetic field is parallel to the slot direction, and couple to the TE mode of the waveguides. On the other hand, magnetic field perpendicular to the slot direction cannot enter the slot. Applying this principle, it is possible to read out the desired di-pole mode, as shown in Figure 2.4. The slots aligned in Y direction are for X mode read out, and those aligned in X direction are for Y mode read out. For example, Y di-pole mode can be read out through the Y slot since its magnetic field is parallel to the Y slot direction, while X di-pole mode cannot be read out since its magnetic field is perpendicular to them. Monopole mode cannot be read out because its magnetic field is also perpendicular to both Y slots and X slots.

Also, noise modes are rejected due to the cut off frequency of the wave guides. By designing the cut off to be under the desired di-pole mode and over the mono-pole or di-pole mode of the other direction, we could select the desired di-pole mode to be read out from the waveguides.

There are two ports for each X di-pole mode read out and Y di-pole mode read out. The signals transmitting from the cavity to the waveguides and out through the antennas are shown in Figure 2.4. The antennas are aligned to be rotational symmetric, so that the phase of E field flip at the 2 ports. Due to this scheme, the signals read out from the 2 ports are opposite in phase.
3 Parameters of Cavity BPMs

In this section, essential parameters for RF cavities are defined. They are Quality Factor $Q$, Coupling Constant $\beta$, and what called $R/Q$ ($R$ over $Q$). Position sensitivity or resolution of cavity BPMs are determined from these parameters.

3.1 Quality Factor $Q$ and Coupling Constant $\beta$

Quality factor $Q$ represents the efficiency of resonant mode of a cavity. The higher $Q$ of a mode is, the lower energy loss would be at the mode, which enables to oscillate effectively. Also at high $Q$ mode, peak of the resonance is sharp since power efficiency difference is large between on-peak and off-peak.

Given that energy $U$ is stored in a cavity for an eigenmode, and power $P$ is lost per angular oscillation of the mode, Quality Factor $Q_L$ (loaded $Q$) is defined as

$$Q_L = \frac{\omega U}{P}, \quad (3.1)$$

where $\omega$ is the angular frequency of the eigenmode. Dissipation power of the cavity, $P$, can be divided to the power which is converted to thermal energy at the inner wall of the cavity, $P_{\text{wall}}$, and the power output from ports, $P_{\text{out}}$. Thus it is written as

$$P = P_{\text{wall}} + P_{\text{out}}. \quad (3.2)$$

Therefore, $Q_L$ is a quality factor including external circuit associated to the considering cavity. Corresponding to $P_{\text{wall}}$ and $P_{\text{out}}$, $Q_0$ and $Q_{\text{ext}}$ are defined as the following:

$$Q_0 = \frac{\omega U}{P_{\text{wall}}}, \quad (3.3)$$
$$Q_{\text{ext}} = \frac{\omega U}{P_{\text{out}}}. \quad (3.4)$$

$Q_0$ is the Quality Factor of the cavity itself, and $Q_{\text{ext}}$ is the Quality Factor of the external coupling. $Q_0$ depends on cavity material and its surface condition, while $Q_{\text{ext}}$ depends not on them but on cavity shape. Thererore $Q_{\text{ext}}$ is preferred to evaluate the cavity performance. From $(3.1) \sim (3.4)$,

$$\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_{\text{ext}}} \quad (3.5)$$

is easily obtained.

The strength of the coupling between the cavity and external circuit, which is the efficiency of extracting energy from the cavity, is called the coupling constant $\beta$ and is defined as

$$\beta \equiv \frac{P_{\text{out}}}{P_{\text{wall}}} = \frac{Q_0}{Q_{\text{ext}}}. \quad (3.6)$$

Therefore in order to improve $\beta$, high $Q_0$ (or low $P_{\text{wall}}$) and low $Q_{\text{ext}}$ (or high $P_{\text{out}}$) is required. Using $\beta$, $Q_0$ and $Q_{\text{ext}}$ are represented as

$$Q_0 = (1 + \beta)Q_L \quad (3.7)$$
$$Q_{\text{ext}} = \frac{Q_0}{\beta}. \quad (3.8)$$

These parameters are mainly measured by a network analyzer, which is discussed in section 6.

By definition, $P$ is also represented as

$$P = -\frac{dU}{dt}. \quad (3.9)$$
Applying (3.9) to (3.11), one can calculate
\[
\frac{dU}{dt} = -\frac{\omega}{Q_L} U \\
U = U_0 e^{-\frac{Q_L}{\omega} t},
\]  
(3.10)

where \( U_0 \) is the initial energy stored in the cavity. Defining decay time constant \( \tau \) as the time length for the stored energy to reduce to \( 1/e \), it is written as
\[
\tau = \frac{Q_L}{\omega} = \frac{Q_L}{2\pi f}.
\]  
(3.11)

Since signal amplitude is proportional to the square root of stored energy, time length for the amplitude to reduce to \( 1/e \) is \( 2\tau \).

3.2 \( R/Q \)

\( R/Q \) is a reference index often used to evaluate the accelerating performance or the effect from beam at a cavity. It depends only on cavity shape, and not on material or surface condition of the cavity.

\( R \) is what called the shunt impedance, which represents the accelerating efficiency of the cavity, and is defined as
\[
R = \left| \frac{P_{\text{wall}}}{E_0} \right|^2, 
\]  
(3.12)

where \( E \) is the accelerating electric field, \( \int ds \) is the integration along beam orbit, and \( P_{\text{wall}} \) is the unit dissipation power of the cavity, as shown in (3.1).

By calculating \( (3.2) \div (3.3) \), it is easily obtained that
\[
\frac{R}{Q} = \frac{P_{\text{wall}}}{\omega_0 U} = \frac{\int E ds}{\omega_0 U} = \frac{|V|^2}{\omega_0 U}. 
\]  
(3.13)

At (3.13), dissipation power of the cavity \( P_{\text{wall}} \) disappears, which means that \( R/Q \) does not depend on material or surface condition of the cavity, and one can evaluate the accelerating field considering only the cavity shape. Considering definition of shunt impedance \( R \), the Quality Factor represented in \( R/Q \) is \( Q_0 \), however it is not represented explicitly by convention. \( R/Q \) of a cavity is usually measured by bead perturbation measurement, which is discussed later in section 6.

Considering time dependency, the beam direction component of electric field \( E_z \) excited at the cavity is represented as
\[
E_z = E_0 \exp(i\omega t). 
\]  
(3.14)

This implies that phase of the field shifts while the beam passes the cavity in a finite time. Considering this effect, induced voltage \( V \) in a cavity of length \( L \) is calculated as
\[
V = \int_{-L/2}^{L/2} E_0 \exp(i\omega t) ds = \int_{-L/2c}^{L/2c} E_0 \exp(i\omega t) c dt = TLE_0, 
\]  
(3.15)

where effect of the phase transition during the beam passing time is represented in \( T \), defined as
\[
T = \frac{\sin(\omega L/2c)}{\omega L/2c}. 
\]  
(3.16)

\( T \) is called the transit time factor.
4 Calculation of output signals

In this section, output signal from a cavity is calculated. First, a beam induced voltage at a cavity is calculated from the "fundamental theorem of beam excitation". Applying this theory, output amplitude of a particular resonant mode of a cavity is calculated. The key factor of output amplitude is $R/Q$, which determine the beam position dependency of the signal. Thus $R/Q$ of simple cavities, such as cylindrical cavity and rectangular cavity are calculated, and their position dependencies are shown. Also, noise components such as beam angle signals and bunch tilt signals are calculated, and their phase difference from position signals are discussed. Therefore at the end of this section, importance of phase detection at beam position monitoring is noticed. Finally, quality factor of a cavity, $Q_0$, is calculated since it is also a parameter comparable with measurement results.

4.1 Fundamental Theorem of beam excitation

Voltage of a resonant mode excited at a cavity by passing beam, $V_{exc}$, is represented as

$$V_{exc} = \frac{\omega}{2} \left( \frac{R}{Q} \right) q,$$

(4.1)

where $\omega$ is the resonant angular frequency of the mode, and $q$ is the beam charge. (4.1) is obtained from what called ”fundamental theorem of beam loading”[6].

Considering a charged particle traveling through free space in speed of light, electro-magnetic field excited by the particle is trapped in a plane transverse to the beam direction. This is the same in case of electron beam bunch traveling in axis direction through a beam pipe of a uniform shaped, perfect conductor. However when the beam pipe has finite impedance or is not uniform, the excited field would remain after the particle has passed. This is called the wake field. Not only the following bunch, but the bunch which excited the field itself is also decelerated by the wake field.

Fundamental theorem of beam loading shows that the electric field which the bunch itself feels is 1/2 of the amplitude and same phase with wake field the considering bunch excites. The beam bunch is decelerated by this wake field, and its energy is lost at the cavity. This theorem is proved by the following discussion.

Let’s assume that initially there is no electric field at the cavity. Then a beam bunch of charge $q$ passes the cavity and excites $V_{exc}$. At the same time, the bunch itself feels $V_{eff}$. The beam bunch is decelerated by $V_{eff}$. To relate $V_{eff}$ to $V_{exc}$, let’s define the amplitude ratio and phase difference between them as $\gamma$ and $\epsilon$, respectively. Therefore they are represented as

$$V_{exc} = -V_{exc}e^{i\epsilon}$$

$$V_{eff} = -\gamma V_{exc},$$

(4.2)

where ”−” represents the decelerating effect the bunch feels.

Next, second bunch is considered. At passing the cavity the second bunch feels the wake field of the first bunch, and also the field which was excited by the second bunch itself. If the phase transition of $V_{exc}$ between the timing of the first and the second is $\theta$ and assuming no energy loss at the cavity, the field $i$ th bunch feels $V_i$ are

$$V_1 = V_{eff}$$

$$V_2 = V_{exc}e^{i\theta} + V_{eff}.$$

(4.3)

Therefore sum of acquired energy of the two bunches, $\Delta U_{beam}$, is calculated as

$$\Delta U_{beam} = q\text{Re}(V_1) + q\text{Re}(V_2) = -2q\gamma V_{exc} - qV_{exc}\cos(\theta + \epsilon).$$

(4.4)

Cavity voltage after the two bunches have passed, $V_{cavity}$, is the sum of the voltage excited from both bunches, which is

$$V_{cavity} = V_{exc}e^{i\theta} + V_{exc}.$$

(4.5)
Also, stored energy of the cavity $\Delta U_{\text{cavity}}$ is calculated from (3.13), which is

$$
\Delta U_{\text{cavity}} = \frac{|V_{\text{cavity}}|^2}{\omega(R/Q)} = \frac{|V_{\text{exc}}e^{i(\epsilon\theta)} + 1|^2}{\omega(R/Q)} = \frac{2V_{\text{exc}}^2(1 + \cos\theta)}{\omega(R/Q)}.
$$

(4.6)

Beam bunches are decelerated by $\Delta U_{\text{cavity}}$. From energy conservation law, lost beam energy at the cavity $-\Delta U_{\text{beam}}$ and $\Delta U_{\text{cavity}}$ must be equivalent. Therefore,

$$
-\Delta U_{\text{beam}} = \Delta U_{\text{cavity}}
$$

$$
\rightarrow \frac{2V_{\text{exc}}^2(1 + \cos)}{\omega(R/Q)} = 2q\gamma V_{\text{exc}} + qV_{\text{exc}}\cos(\theta + \epsilon)
$$

$$
\rightarrow \left( \frac{2V_{\text{exc}}^2}{\omega(R/Q)} - 2q\gamma V_{\text{exc}} \right) + \left( \frac{2V_{\text{exc}}^2}{\omega(R/Q)} - qV_{\text{exc}}\cos\epsilon \right) \cos\theta + qV_{\text{exc}}\sin\epsilon\sin\theta = 0.
$$

(4.7)

Of course $\theta$ varies arbitrarily, since bunch interval can be changed arbitrarily. Therefore identical equation (4.7) must work out in no relation to $\theta$. This leads to

$$
\frac{2V_{\text{exc}}^2}{\omega(R/Q)} - 2q\gamma V_{\text{exc}} = 0
$$

(4.8)

$$
\frac{2V_{\text{exc}}^2}{\omega(R/Q)} - qV_{\text{exc}}\cos\epsilon = 0
$$

(4.9)

$$
qV_{\text{exc}}\sin\epsilon = 0.
$$

(4.10)

and from these equations

$$
\epsilon = 0
$$

(4.11)

$$
V_{\text{exc}} = \frac{\omega}{2} \left( \frac{R}{Q} \right) q
$$

(4.12)

$$
\gamma = \frac{1}{2}
$$

(4.13)

are obtained. Therefore, amplitude ratio between $V_{\text{eff}}$ and $V_{\text{exc}}$ is 1/2, and their phase equate.

Fundamental theorem of beam loading is proved, and also (4.1) is proven.

### 4.2 Effect of bunch length

In case of the former section, bunch length in beam direction was not considered. If the bunch length is not negligible against frequency of the excited mode, effect of bunch length must be considered. For each particle consisting the bunch, (4.11) can be applied. However as a total bunch, since each particle passes the cavity in different timing, phase of the excited mode would shift during the bunch passage. Therefore electric field excited by each particle would be summed up in different phase, which reduces the effective signal compared to single particle case.

Let’s assume a gaussian distribution in beam direction with beam size $\sigma_z$,

$$
\frac{1}{\sqrt{2\pi}\sigma_z} \exp \left( -\frac{z^2}{2\sigma_z^2} \right).
$$

(4.14)

Comparing charged particle at $z = 0$ and $z = z'$, the timing difference of passing the cavity $\Delta t$ is $z'/c$. Therefore the phase shift $\Delta \theta$ is $\omega z'/c$. The total excited voltage $V_{\text{totalexc}}$ is obtained by
integrating all the particles in the bunch, which is

\[ V_{\text{totalexc}} = \int_{-\infty}^{\infty} V_{\text{exc}} \frac{1}{\sqrt{2\pi}\sigma_z} \exp \left( -\frac{z^2}{2\sigma_z^2} \right) \frac{1}{2\pi^2} \left( \frac{\omega^2}{c} \sigma_z^2 \right) \exp \left( -\frac{1}{2}\frac{\omega^2}{c} \sigma_z^2 \right) e^{izn} dz \]

Compared with single particle or infinitely small bunch, excited voltage is reduced by factor \( \exp \left( -\frac{\omega^2\sigma_z^2}{2c^2} \right) \). Considering stored energy, the factor is \( \exp \left( -\frac{\omega^2\sigma_z^2}{c^2} \right) \), since energy is the square of amplitude.

## 4.3 Output signals

Finally output signal from the cavity can be calculated. From (4.12) and (4.15), total excited voltage \( V_{\text{totalexc}} \) is represented as

\[ V_{\text{totalexc}} = \frac{\omega}{2} \frac{R}{Q} q^2 \exp \left( -\frac{\omega^2\sigma_z^2}{2c^2} \right). \]  

Applying (4.16) to (3.13), stored energy of a cavity is calculated as

\[ U = \frac{V_{\text{totalexc}}^2}{\omega(R/Q)} = \frac{\omega^2}{4} (R/Q) q^2 \exp \left( -\frac{\omega^2\sigma_z^2}{c^2} \right). \]  

From the definition of \( Q_{\text{ext}} \) (3.4),

\[ P_{\text{out}} = \frac{\omega U}{Q_{\text{ext}}} = \frac{\omega^2 q^2}{4Q_{\text{ext}}} (R/Q) q^2 \exp \left( -\frac{\omega^2\sigma_z^2}{c^2} \right) \]

is obtained. Detecting this power by impedance \( Z \), the output voltage is

\[ V_{\text{out}0} = \sqrt{ZP} = \frac{\omega q}{2} \sqrt{\frac{Z}{Q_{\text{ext}}} (R/Q)} \exp \left( -\frac{\omega^2\sigma_z^2}{2c^2} \right). \]  

Notice that the output voltage is proportional to \( \sqrt{R/Q} \) and \( 1/\sqrt{Q_{\text{ext}}} \). \( Q_{\text{ext}} \) is determined from the cavity design and is a constant for a particular cavity, while \( R/Q \) is a function of beam offset. In case of di-pole mode, \( R/Q \) is proportional to the square offset, and therefore output signal amplitude \( V_{\text{out}0} \) is proportional to the beam offset. In case of mono-pole mode, \( R/Q \) is a constant and therefore is called as “common mode”.

Considering the decay time constant, output signal extracted from the cavity is represented as

\[ V_{\text{out}} = V_{\text{out}0} e^{-\frac{t}{\tau}} \sin(\omega t + \phi), \]

which is shown in Figure 4.1.
4.4 \( R/Q \) of a Cylindrical Cavity

As shown in (4.19), calculation of \( R/Q \) is important to understand the output signal from a cavity. Thus \( R/Q \) is calculated specifically in case of cylindrical cavity in this section, and rectangular cavity in the next section.

As shown in Appendix A.1, the \( \text{TM}_{mnp} \) mode electric field of a cylindrical cavity is represented as

\[
\begin{align*}
E_r &= -E_0 \frac{k_z}{k} J_m(k_c r) \cos(m \phi) \sin \left( \frac{p \pi}{L} z \right) e^{i \omega t} \\
E_\phi &= E_0 \frac{k_z}{k} J_m(k_c r) \sin(m \phi) \sin \left( \frac{p \pi}{L} z \right) e^{i \omega t} \\
E_z &= E_0 J_m(k_c r) \cos(m \phi) \cos \left( \frac{p \pi}{L} z \right) e^{i \omega t},
\end{align*}
\]

(4.21)

where \( L \) is the cavity length in \( z \) direction, and \( m, n, p \) are node numbers in \( \phi, r, z \) direction of the cylindrical coordinate, respectively. \( k_c, k_z \) are wave numbers of \( \phi \) and \( r \) direction, and \( z \) direction respectively. From the boundary condition,

\[
k^2 = k_c^2 + k_z^2
\]

\[
k_c = k_{mn}^c = \frac{j_{mn}}{a}
\]

(4.22)

is obtained, where \( j_{mn} \) is the \( n \)th root of Bessel function \( J_m \), and \( a \) is the diameter of the cylinder.

4.4.1 Di-pole mode

Considering di-pole mode \( \text{TM}_{110} \), the electric field is represented as

\[
\begin{align*}
E_r &= E_0 \cos(\phi) J_1(k_{11} c) e^{i \omega t} \\
E_\phi &= 0 \\
E_z &= E_0 J_1(k_{11} c) e^{i \omega t}.
\end{align*}
\]

(4.23)

Using the dispersion relation \( \omega = ck \) and given \( j_{11} \simeq 3.83 \),

\[
k_{11}^c = \frac{\omega_{11}}{c} = \frac{3.83}{a}
\]

(4.24)

is obtained.

When beam passes the cavity in \( z \) direction with an offset \( x \), electric field excited in the cavity is calculated by applying \( r = x, \phi = 0 \) to (4.23), which is

\[
V(x) = \int_0^L E_z dz = \int_0^L E_0 J_1(k_{11} x) e^{i \omega t} dz
\]
Applying indefinite integral one can calculate and recurrence formula of Bessel function

\[ U = \frac{1}{2} \int_0^{\pi} e^{-\frac{i}{T}} \int_0^{\pi} \cos^2 \phi J_1^2(k_c r) d\phi d\phi \]

where \( T \) is the transit time factor (3.16). The stored energy \( U \) is calculated as

\[ U = \frac{1}{2} \int_0^{\pi} |E_z|^2 d\phi = \frac{1}{2} \int_0^{\pi} \cos^2 \phi J_1^2(k_c r) d\phi \]

Applying indefinite integral

\[ \int J_n^2(k r) rdr = \frac{r^2}{2} \left( J_n^2(k r) - J_{n-1}(k r) J_{n+1}(k r) \right) \]

and recurrence formula of Bessel function

\[ J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x), \]

one can calculate

\[ \int_0^a J_1^2(k_c r) rdr = \left[ \frac{r^2}{2} \left( J_1^2(k_c r) - J_0(k_c r) J_2(k_c r) \right) \right]_0^a \]

\[ = \frac{a^2}{2} \left[ J_1^2(k_c a) - J_0(k_c a) J_2(k_c a) \right] \]

\[ = -\frac{a^2}{2} J_0(k_c a) J_2(k_c a) \quad \text{(from } ak_1 = j_{11}, J_1(j_{11}) = 0) \]

\[ = -\frac{a^2}{2} J_0(k_c a) \left[ \frac{2}{k_c} J_1(k_c a) - J_0(k_c a) \right] \quad \text{(from (4.28))} \]

\[ = \frac{a^2}{2} J_0^2(k_c a). \]
is obtained. In addition by applying approximate expression of Bessel Function

\[ J_n(x) \approx \frac{1}{n!} \left( \frac{x}{2} \right)^n \quad (x \approx 0) \]  

(4.32)

and considering the case of \( x \approx 0 \),

\[ J_1(k_{c1}^1 x) \approx \frac{1}{2} k_{c1}^1 x = \frac{1}{2} \omega \frac{x}{c} \]  

(4.33)

is obtained. Applying (4.33) and

\[ J_0(k_{c1}^1 \alpha) = J_0(3.83) \approx -0.403, \]  

(4.34)

\( k \) and \( \alpha \) is canceled in (4.31), which leads to

\[ \frac{R}{Q} = \frac{1}{\times 3.83^2 \times 0.403^2 \times c \times \epsilon_0 \times \pi \left( \frac{\omega}{c} \right)^3 L T^2 x^2} \]

\[ \approx 50.3 \times \left( \frac{\omega}{c} \right)^3 L T^2 x^2. \]  

(4.35)

(4.35) implies that \( R/Q \) is proportional to \( x^2 \), and considering output amplitude \( V_{out0} \propto \sqrt{R/Q} \), \( V_{out0} \) is proportional to \( x \).

### 4.4.2 Mono-pole mode

Mono-pole mode \( TM_{010} \) is often used for beam acceleration at accelerating cavities. Its electric field is represented as

\[ \begin{align*}
E_r &= E_\phi = 0 \\
E_z &= E_0 J_0(k_{c1}^{01} r) e^{i \omega t}.
\end{align*} \]  

(4.36)

Given that \( j_{01} \approx 2.41 \),

\[ k_{c1}^{01} = \frac{\omega_{01}}{c} = \frac{2.41}{a} \]  

(4.37)

is obtained.

As is same with the di-pole mode calculation, the excited electric field in a cavity by the passing beam is calculated as

\[ V(x) = \int_0^L E_z dz = \int_0^L E_0 J_0(k_{c1}^{01} x) e^{i \omega t} dz \]

\[ = E_0 J_0(k_{c1}^{01} x) L T e^{i \frac{\omega t}{2}}. \]  

(4.38)

Stored energy of the cavity \( U \) is calculated as

\[ U = \frac{1}{2} \int \epsilon_0 |E_z|^2 dV = \frac{1}{2} \epsilon_0 E_0^2 \int J_0^2(k_{c1}^{01} r) dV \]

\[ = \frac{1}{2} \epsilon_0 E_0^2 \int_0^L \int_0^{2 \pi} \int_0^a \cos^2 \phi J_1^2(k_{c1}^{11} r) r dr d\phi dz \]

\[ = \frac{1}{2} \epsilon_0 E_0^2 \pi L \int_0^a J_1^2(k_{c1}^{11} r) r dr. \]  

(4.39)
Applying indefinite integral
\[
\int J_0^2(kr)dr = \frac{r^2}{2} \left( J_0^2(kr) + J_1^2(kr) \right),
\]
\[4.40\]
\[
\text{is calculated as}
\int_0^a J_0^2(k_{c01}r)dr = \left[ \frac{r^2}{2} \left( J_0^2(k_{c01}a) + J_1^2(k_{c01}a) \right) \right]_0^a = \frac{a^2}{2} \left( J_0^2(k_{c01}a) + J_1^2(k_{c01}a) \right) \quad \text{(from } ak_{c01} = j_{01}, J_0(j_{01}) = 0).\]
\[4.41\]
Applying \[4.41\], \[4.39\] is represented as
\[
U = \frac{1}{2} \varepsilon_0 E_0^2 \pi L a^2 J_1^2(k_{c01}a).\]
\[4.42\]
Applying \[4.38\] and \[4.42\] to \[3.13\],
\[
\frac{R}{Q} = \frac{|V|^2}{\omega U} = \frac{E_0^2 J_0^2(k_{c01}x)L^2T^2}{\omega \pi \varepsilon_0 E_0^2 a^2 J_1^2(k_{c01}a)} = \frac{2J_0^2(k_{c01}x)L^2T^2}{\omega \varepsilon_0 \pi J_1^2(k_{c01}a)a^2} \quad \text{(4.43)}
\]
is obtained. In addition by applying approximate expression of Bessel Function \[4.32\] and assuming \(x \approx 0\),
\[
\frac{R}{Q} = \frac{2LT^2}{\omega \varepsilon_0 \pi J_1^2(k_{c01}R)R^2} \quad \text{(4.44)}
\]
is obtained.

Therefore it is shown that \(R/Q\) of mono-pole mode is not sensitive to beam position near the cavity center. In this case beam position dependency of output signal would disappear, and would be sensitive to beam charge only.

### 4.5 \(R/Q\) of a Rectangular Cavity

As shown in Appendix A.1 the TM\(_{mnl}\) mode electric field of a rectangular cavity is represented as
\[
E_z = E_0 \sin \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) \cos \left( \frac{l\pi}{L} z \right) e^{i\omega t},
\]
\[
E_x = -\frac{k_z}{k^2 - k_z^2} E_0 \frac{m\pi}{a} \cos \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) \sin \left( \frac{l\pi}{L} z \right) e^{i\omega t},
\]
\[
E_y = -\frac{k_z}{k^2 - k_z^2} E_0 \frac{n\pi}{b} \sin \left( \frac{m\pi}{a} x \right) \cos \left( \frac{n\pi}{b} y \right) \sin \left( \frac{l\pi}{L} z \right) e^{i\omega t},
\]
where \(a, b, L\) are the length of the cavity in \(x, y\) and \(z\) direction, respectively.
4.5.1 Di-pole mode

There are two di-pole modes for a rectangular cavity, which are X di-pole mode $TM_{210}$ and Y di-pole mode $TM_{120}$, as discussed in section \[2\]. Considering Y di-pole mode $TM_{120}$, the electric field is represented as

$$E_x = E_y = 0 \quad E_z = E_0 \sin \left( \frac{\pi}{a} x \right) \sin \left( \frac{2\pi}{b} y \right) e^{i\omega t}. \quad (4.46)$$

When beam passes the cavity in $z$ direction, electric field excited in the cavity is

$$V = \int_0^L E_z dz = E_0 \sin \left( \frac{\pi}{a} x \right) \sin \left( \frac{2\pi}{b} y \right) \int_0^L e^{i\omega t} \, dz$$

$$= E_0 \sin \left( \frac{\pi}{a} x \right) \sin \left( \frac{2\pi}{b} y \right) \frac{c}{i\omega} \left[ e^{i\omega L} - 1 \right]$$

$$= E_0 \sin \left( \frac{\pi}{a} x \right) \sin \left( \frac{2\pi}{b} y \right) \frac{c}{i\omega} \left[ e^{i\frac{\omega L}{c}} - e^{-i\frac{\omega L}{c}} \right]$$

$$= E_0 \sin \left( \frac{\pi}{a} x \right) \sin \left( \frac{2\pi}{b} y \right) L T e^{i\frac{\omega L}{c}}, \quad (4.47)$$

where $T$ is the transit time factor \[3.10\]. On the other hand, stored energy of the cavity $U$ is calculated as

$$U = \frac{1}{2} \epsilon_0 \int |E_z|^2 dV = \frac{1}{2} \epsilon_0 \int_0^a \int_0^b \int_0^L E_0^2 \sin^2 \left( \frac{\pi}{a} x \right) \sin^2 \left( \frac{2\pi}{b} y \right) \, dx \, dy \, dz$$

$$= \frac{1}{8} \epsilon_0 E_0^2 ab L. \quad (4.48)$$

They will give $R/Q$ as

$$\frac{R}{Q} = \frac{|V|^2}{\omega U} = \frac{E_0^2 \sin^2 \left( \frac{\pi}{a} x \right) \sin^2 \left( \frac{2\pi}{b} y \right) L^2 T^2}{\omega \epsilon_0 E_0^2 ab L}$$

$$= \frac{8LT^2}{\omega \epsilon_0 ab} \sin^2 \left( \frac{\pi}{a} x \right) \sin^2 \left( \frac{2\pi}{b} y \right). \quad (4.49)$$

Given that beam passes the $x$ center, by applying $x = a/2$ to \[4.49\],

$$\frac{R}{Q} (y) = \frac{8LT^2}{\omega \epsilon_0 ab} \sin^2 \left( \frac{2\pi}{b} y \right) \quad (4.50)$$

is obtained. In case of beam passing near the cavity center, by applying $y = b/2 + y'$ ($y' \approx 0$) to \[4.50\],

$$\frac{R}{Q} (y') = \frac{8LT^2}{\omega \epsilon_0 ab} \sin^2 \left( \frac{2\pi}{b} (y' + \pi) \right) \approx \frac{8LT^2}{\omega \epsilon_0 ab} \sin^2 \left( \frac{2\pi}{b} y' \right) \quad (4.51)$$

is obtained. Regarding $Y$ offset from the center $y'$ as $y$,

$$\frac{R}{Q} (y) = \frac{8LT^2}{\omega \epsilon_0 ab} \left( \frac{2\pi}{b} \right)^2 y^2 \quad (4.52)$$

is easily shown. \[4.52\] implies that $R/Q$ is proportional to square of beam offset $y^2$, and considering $V_{out0} \propto \sqrt{R/Q}$, $V_{out0}$ is proportional to offset $y$. 

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4 CALCULATION OF OUTPUT SIGNALS

20
4.5.2 Mono-pole mode

From (4.45), electric field of $TM_{110}$ is represented as

$$
E_z = E_0 \sin \left( \frac{\pi}{a} x \right) \sin \left( \frac{\pi}{b} y \right) e^{i\omega t} \tag{4.53}
$$
$$
E_x = E_y = 0. \tag{4.54}
$$

After calculation similar as di-pole mode, electric field excited along the beam orbit is obtained as

$$
V = \int_0^L E_z dz = E_0 \sin \left( \frac{\pi}{a} x \right) \sin \left( \frac{\pi}{b} y \right) \int_0^L e^{i\omega t} dz
$$
$$
= E_0 \sin \left( \frac{\pi}{a} x \right) \sin \left( \frac{\pi}{b} y \right) L T e^{i \frac{\pi z}{2L}}. \tag{4.55}
$$

Stored energy of the cavity $U$ is

$$
U = \frac{1}{2} \varepsilon_0 \int |E_z|^2 dV = \frac{1}{2} \varepsilon_0 \int_0^a \int_0^b \int_0^L E_0^2 \sin^2 \left( \frac{\pi}{a} x \right) \sin^2 \left( \frac{\pi}{b} y \right) dxdydz
$$
$$
= \frac{1}{8} \varepsilon_0 E_0^2 abL. \tag{4.56}
$$

Therefore, R/Q is calculated as

$$
\frac{R}{Q} = \frac{|V|^2}{\omega U} = \frac{E_0^2 \sin^2 \left( \frac{\pi}{a} x \right) \sin^2 \left( \frac{\pi}{b} y \right) L^2 T^2}{\omega \frac{1}{8} \varepsilon_0 E_0^2 abL}
$$
$$
= \frac{8LT^2}{\omega \varepsilon_0 ab} \sin^2 \left( \frac{\pi}{a} x \right) \sin^2 \left( \frac{\pi}{b} y \right). \tag{4.57}
$$

Considering X axis and applying $y = b/2$, (4.57) is calculated as

$$
\frac{R}{Q}(x) = \frac{8LT^2}{\omega \varepsilon_0 ab} \sin^2 \left( \frac{\pi}{a} x \right). \tag{4.58}
$$

Moreover, assuming the beam passing near the cavity center, $x = a/2 + x' \ (x' \approx 0)$ is applied. Thus,

$$
\frac{R}{Q}(x') = \frac{4LT^2}{\omega \varepsilon_0 ab} \sin^2 \left( \frac{\pi}{a} \left( \frac{a}{2} + x' \right) \right) = \frac{4LT^2}{\omega \varepsilon_0 ab} \sin^2 \left( \frac{\pi x'}{a} + \frac{\pi}{2} \right)
$$
$$
= \frac{4LT^2}{\omega \varepsilon_0 ab} \cos^2 \left( \frac{\pi x'}{a} \right). \tag{4.59}
$$

is obtained. As a result, mono-pole mode is not sensitive to beam position near the cavity center. Therefore mono-pole mode is used to monitor beam charge, independent of beam position.

4.6 Noise Components

To acquire di-pole mode signal, frequency selection using band pass filters, and spatial couplers as discussed in section 2 are applied. As a result, output signals would be represented as

$$
V = V_{\text{position}} + iV_{\text{angle}} + iV_{\text{tilt}} + iV_{\text{common tail}}. \tag{4.60}
$$

$V_{\text{position}}$ is the desired position signal, while others are the noise components. $V_{\text{angle}}$ originates from beam orbit angle, $V_{\text{tilt}}$ originates from bunch tilt, and $V_{\text{common tail}}$ is the tail of common mode signal[7]. Term $i$ shows $\pi/2$ phase shift. As discussed in the following sections, the noise components are $\pi/2$ different in phase compared with position signal. Therefore, phase detection is essential for beam position monitoring.
4.6.1 Beam Angle Signals

When beam orbit is tilted against the cavity, it excites di-pole mode even when it passes the cavity center. This is mainly due to phase transition during the beam passing time.

Defining cavity length $L$ and resonant frequency of the di-pole mode $\omega$, output power from the cavity in case of non-tilted beam passing the cavity with an offset $x$ is represented as

$$U \propto \frac{R}{Q} \propto L x^2,$$

(4.61)
as is shown from (4.35) and (4.52). Therefore, the output amplitude is proportional to $\sqrt{L} x$. Defining an appropriate coefficient $A$, beam position signal is represented as

$$V_{\text{position}} = Ax \sqrt{L} \sin(\omega t).$$

(4.62)

Next, a beam passing the cavity center with an angle $x'$ is considered. Assuming that the angle $x'$ is small enough ($\tan x' \sim x'$), it can be modeled as Figure 4.2. Namely, the angle signal is regarded as sum of two position signals. One is entering the cavity of length $L/2$ with $-x'L/4$ offset, while the other is entering the cavity of length $L/2$ with $x'L/4$ offset. Considering $\pm L/4c$ phase difference between the two signals, their sum is calculated as

$$V_{\text{angle}} = -Ax' \frac{L}{4} \sqrt{\frac{L}{2}} \sin \left( \omega \left( t + \frac{L}{4c} \right) \right) + Ax' \frac{L}{4} \sqrt{\frac{L}{2}} \sin \left( \omega \left( t - \frac{L}{4c} \right) \right)$$

$$= Ax' \frac{L}{2} \sqrt{\frac{L}{2}} \sin \left( \frac{\omega L}{4c} \right) \cos(\omega t).$$

(4.63)

From (4.62) and (4.63), the amplitude ratio of angle signal to position signal is

$$\frac{|V_{\text{angle}}|}{|V_{\text{position}}|} = \frac{Ax' \frac{L}{2} \sqrt{\frac{L}{2}} \sin \left( \frac{\omega L}{4c} \right)}{Ax \sqrt{L}}$$

$$= \frac{L}{2 \sqrt{2}} \sin \left( \frac{\omega L}{4c} \right) \frac{x'}{x} \approx \frac{\omega L^2 x'}{8 \sqrt{2}c x}.$$

(4.64)

(4.64) implies that the angle sensitivity of a cavity is proportional to the square of cavity length, $L^2$. Also, it is shown that the angle signal and the position signal are 90 degrees different in phase.

Figure 4.2: Modeling of beam angle signal
4.6.2 Bunch Tilt Signals

Considering finite bunch length, tilted beam bunch excites di-pole mode even when it passes the cavity center in correct beam orbit. It also originates from the phase transition of the passing beam, and can also be explained by a simple model.

Since beam position signal is proportional to beam charge \( q \) and offset \( x \), the position signal can be represented as

\[
V_{\text{position}} = Aqx\sin(\omega t),
\]

(4.65)

where \( A \) is the proportionality coefficient and \( \omega \) is the resonant angular frequency of the mode. As shown in Figure 4.3, tilted beam bunch of bunch length \( \sigma_z \) can be modeled as two point charges with charge of \( q/2 \). The two point charges are located at distance of \( \sigma_z \) from the beam center in \( z \) direction. \( V_+, V_- \) are defined as the signals excited by each point charge. Considering phase difference at passing the cavity and applying beam charge \( q/2 \) and beam offset \( \pm \sigma_z \tan\theta \simeq \pm \sigma_z \theta \) to (4.65),

\[
V_+ = A\frac{q}{2}\sigma_z\theta\sin \left( \omega \left( t + \frac{\sigma_z}{c} \right) \right)
\]

(4.66)

\[
V_- = A\frac{q}{2}\left(-\sigma_z \theta\right)\sin \left( \omega \left( t - \frac{\sigma_z}{c} \right) \right)
\]

(4.67)

are obtained. Since the excited signal \( V_{\text{tilt}} \) would be the sum of these signals,

\[
V_{\text{tilt}} = V_+ + V_- = Aq\sigma_z\theta\sin \left( \frac{\omega \sigma_z}{c} \right) \cos(\omega t)
\]

\[
\simeq Aq\theta\omega \frac{\sigma_z^2}{c} \cos(\omega t)
\]

(4.68)

is obtained. As shown in (4.68), bunch tilt signal is proportional to tilt angle \( \theta \) and square of bunch length \( \sigma_z^2 \). Also, it is shown that bunch tilt signal is 90 degrees different in phase compared to position signal.

![Figure 4.3: Modeling of tilted bunch signal](image)

4.6.3 Common Mode Signals

Since mono-pole mode does not have a node at cavity center and therefore signal is almost flat, as shown in (4.59), beam always excites the same voltage for mono-pole mode. Thus mono-pole mode is called as "common mode". Common mode excitation is so strong that its tail is usually still dominant at frequency of di-pole modes. To avoid its contamination, common mode signal is cut off by a spatial coupler as discussed in section 2.4 and is also rejected by phase detection.

As described later in section 6, excited power at resonant frequency \( \omega_0 \) is represented as

\[
E(\omega) = \frac{E_0}{\sqrt{2\pi}} \frac{1}{\omega_0^2 Q_L^2 + \omega^2} \left( \omega_0 - \omega \right)
\]

(4.69)
as a function of angular frequency $\omega$. After a simple calculation, its normalized amplitude $A$ and phase $\theta$ are obtained as

$$A = \frac{\omega_0}{\sqrt{Q_L}} \left\{ \sqrt{\left(\frac{\omega_0}{Q_L}\right)^2 + (\omega - \omega_0)^2} \right\}$$

(4.70)

$$\theta = \tan^{-1}\left(-2\frac{\omega - \omega_0}{\omega_0 Q_L}\right).$$

(4.71)

They are shown in Figure 4.4. In Figure 4.4, amplitude of desired Y di-pole mode is also shown. Both modes are calculated assuming 61.40 mm × 48.56 mm × 6 mm rectangular cavity, similar to the IP-BPM cavity.

As shown in (4.71), common mode signal at off peak is $\pi/2$ different in phase compared with peak frequency.

![Figure 4.4: Tail of common mode signal](image)

### 4.7 Calculation of $Q_0$

As described in section 3, $Q_{ext}$ is a factor which doesn’t depend on cavity material or surface condition. However, $Q_{ext}$ can be affected by the external coupling, for example incomplete antenna coupling due to fabrication accuracy. On the other hand, $Q_0$ is ideally determined from cavity material and its shape, which is analytically calculated in case of simple geometries. Therefore, $Q_0$ can be used to compare with measurement results and evaluate the surface condition.

#### 4.7.1 Calculation of energy loss

In order to calculate $Q_0$, energy loss at the cavity wall $P_{wall}$ must be calculated. Defining surface resistance and current of the inner wall as $R_s$ and $I_s$ respectively, $P_{wall}$ is represented as

$$P_{wall} = \int_S R_s I_s^2 dS = \frac{1}{2} R_s \int_S |H_/|^2 dS,$$

(4.72)
where $S$ is the cavity surface, and $H_{//}$ is the magnetic field parallel to $S$. In (4.72), $R_s$ is assumed to be constant over $S$, and also
\[ R_s I_s^2 = \frac{1}{2} R_s |H_{//}|^2 \] (4.73)
is used. (4.73) is derived at Appendix A.2.

In case of perfect conductive surface, there is no electric field in the conductor. However in practice, there is no such infinite conductivity. Therefore, electric field parallel to the cavity surface, $E_{//}$, penetrates into the conductor. Skin depth $\delta$ of a material is defined as the depth where amplitude of $E_{//}$ becomes $1/e$ compared with amplitude at the surface. It is represented as
\[ \delta = \sqrt{\frac{2}{\omega \sigma \mu}}, \] (4.74)
where $\omega$ is the considering angular frequency of the field, $\sigma$ is the conductivity of the material, and $\mu$ is the magnetic permeability of the material. From $\delta$ and $\sigma$, $R_s$ is represented as
\[ R_s = \frac{1}{\delta \sigma} = \sqrt{\frac{\omega \mu}{2 \sigma}}. \] (4.75)

Applying (4.75) to (4.72), $P_{wall}$ of a cavity is obtained.

4.7.2 Cylindrical Cavity Case

First, mono-pole mode of a cylindrical cavity is considered, since it is applied at reference cavity of IP-BPM, which is introduced in section 5. Also, it is the most simple mode to calculate.

Let’s consider a cylindrical cavity of radius $a$, and length $L$ in $z$ direction. As shown in Appendix A.1, magnetic filed of TM$_{010}$ is represented as
\[ H_r = H_z = 0 \] (4.76)
\[ H_\phi = -i \frac{\omega \epsilon_0}{k_c} E_0 J_0(k_c r) e^{i \omega t}, \] (4.77)
by applying $m = 0$, $n = 1$, and $l = 0$ to (A.89) $\sim$ (A.91). In this case,
\[ k_c = \frac{j_{01}}{a} = \frac{2.41}{a} \] (4.78)
is applied.

First, lateral surface $S_1$, which is represented as $r = a$, of the cavity is considered. Since (4.77) is a function of $r$,
\[ \int_{S_1} |H_{//}|^2 dS = \int_{S_1} |H_\phi|^2 dS = \left( \frac{\omega \epsilon_0}{k_c} \right)^2 E_0^2 \cdot 2 \pi a L J_0^2(k_c r) \]
\[ = \left( \frac{\omega \epsilon_0}{k_c} \right)^2 E_0^2 \cdot 2 \pi a L J_0^2(k_c r) \] (4.79)
is obtained. Please note that formula of Bessel function
\[ J_0(x) = -J_1(x) \] (4.80)
was used.
Nest, basal surface $S_2$, which is represented as $z = 0$, is considered. Applying (4.27) and (4.80),

$$\int_{S_2} |H/|^2 \, dS = \int_{S_2} |H_0|^2 \, dS = \left(\frac{\omega \varepsilon_0}{k_c}\right)^2 E_0^2 \cdot 2\pi \int_0^a r J_0^2(k_c r) \, dr$$

$$= \left(\frac{\omega \varepsilon_0}{k_c}\right)^2 E_0^2 \cdot 2\pi \int_0^a r J_2^2(k_c r) \, dr$$

$$= \left(\frac{\omega \varepsilon_0}{k_c}\right)^2 E_0^2 \cdot 2\pi \int_0^a \left[ \frac{r^2}{2} \left( J_0^2(k_c r) - J_0(k_c r)J_2(k_c r) \right) \right]_0^a$$

$$= \left(\frac{\omega \varepsilon_0}{k_c}\right)^2 E_0^2 \cdot 2\pi \cdot \frac{a^2}{2} J_2^2(j_01) \quad \text{(from} \ a k_c = j_01, \ J_0(0) = 0)$$

(4.81)

is obtained. This is the same case with surface $S_3$, represented as $z = L$. Therefore, the surface integral over cavity surface $S$ is obtained as

$$\int_S |H/|^2 \, dS = \int_{S_1} |H/|^2 \, dS + \int_{S_2} |H/|^2 \, dS + \int_{S_3} |H/|^2 \, dS$$

$$= \left(\frac{\omega \varepsilon_0}{k_c}\right)^2 E_0^2 \cdot 2\pi a(L + a) J_1^2(j_01).$$

(4.82)

Applying (4.39) and (4.82) to (3.3), $Q_0$ is calculated as

$$Q_0 = \frac{\omega U}{\frac{1}{2} R_s \int_S |H/|^2 \, dS} = \frac{L j_01}{2R_s \omega \varepsilon_0 (L + a)a}.$$

(4.83)

4.7.3 Rectangular Cavity Case

Next, di-pole mode of a rectangular cavity is considered, since it is applied at sensor cavity of IP-BPM, which is introduced in section 5. Let’s consider a cylindrical cavity of length $a$, $b$ and $L$ in $x$, $y$, and $z$ direction. As shown in Appendix A.1, magnetic filed of Y di-pole mode $TM_{120}$ is represented as

$$H_x = i \frac{\omega \varepsilon_0}{k^2 - k_z^2} E_0 \frac{2\pi}{b} \sin \left( \frac{\pi}{a} x \right) \cos \left( \frac{2\pi}{b} y \right) e^{i\omega t}$$

(4.84)

$$H_y = -i \frac{\omega \varepsilon_0}{k^2 - k_z^2} E_0 \frac{\pi}{a} \cos \left( \frac{\pi}{a} x \right) \sin \left( \frac{2\pi}{b} y \right) e^{i\omega t}$$

(4.85)

$$H_z = 0,$$

(4.86)

by applying $m = 1$, $n = 2$, and $l = 0$ to (A.49) $\sim$ (A.51). In this case,

$$k^2 - k_z^2 = \left( \frac{\pi}{a} \right)^2 + \left( \frac{2\pi}{b} \right)^2$$

(4.87)

is applied.

First, $x\rightarrow y$ surface, which is represented as $z = 0$ or $z = L$, is considered. Considering $z = 0$,

$$\int_S |H/|^2 \, dS = \int_S (|H_x|^2 + |H_y|^2) \, dS$$

$$= \left(\frac{\omega \varepsilon_0}{k^2 - k_z^2}\right)^2 E_0^2 \left[ \left( \frac{2\pi}{b} \right)^2 \int_0^a \int_0^b \sin^2 \left( \frac{\pi}{a} x \right) \cos^2 \left( \frac{2\pi}{b} y \right) \, dx \, dy \right]$$

$$+ \left( \frac{\pi}{a} \right)^2 \int_0^a \int_0^b \cos^2 \left( \frac{\pi}{a} x \right) \sin^2 \left( \frac{2\pi}{b} y \right) \, dx \, dy$$

$$= \left(\frac{\omega \varepsilon_0}{k^2 - k_z^2}\right)^2 E_0^2 \left[ \left( \frac{2\pi}{b} \right)^2 + \left( \frac{\pi}{a} \right)^2 \right] \frac{ab}{4}$$

(4.88)
is obtained. Next, $y$-$z$ surface, which is represented as $x = 0$ or $x = a$, is considered. On these surfaces, $\cos\left(\frac{\pi}{a}x\right) = 1$ is applied. Therefore considering $x = 0$,

$$
\int_S \left| H_{//} \right|^2 dS = \int_S \left| H_y \right|^2 dS
$$

$$
= \left( \frac{\omega\epsilon_0}{k^2 - k_z^2} \right)^2 E_0^2 \left( \frac{\pi}{a} \right)^2 \int_0^b \int_0^L \sin^2 \left( \frac{2\pi}{b} y \right) dydz
$$

$$
= \left( \frac{\omega\epsilon_0}{k^2 - k_z^2} \right)^2 E_0^2 \left( \frac{\pi}{a} \right)^2 \frac{bL}{2}
$$

(4.89)

is obtained. In a similar way, on $z$-$x$ surface

$$
\int_S \left| H_{//} \right|^2 dS = \int_S \left| H_x \right|^2 dS
$$

$$
= \left( \frac{\omega\epsilon_0}{k^2 - k_z^2} \right)^2 E_0^2 \left( \frac{2\pi}{b} \right)^2 \frac{aL}{2}
$$

(4.90)

is obtained. Therefore considering the facing surface of each pair, whole surface integral is represented as

$$
\int_S \left| H_{//} \right|^2 dS = 2 \left( \frac{\omega\epsilon_0}{k^2 - k_z^2} \right)^2 E_0^2 \left[ \left( \frac{\pi}{a} \right)^2 \frac{bL}{2} + \left( \frac{2\pi}{b} \right)^2 \frac{aL}{2} + \left( \frac{\pi}{a} \right)^2 + \left( \frac{2\pi}{b} \right)^2 \right] \frac{ab}{4}
$$

(4.91)

Applying (4.48) and (4.91) to (3.3), $Q_0$ is calculated as

$$
Q_0 = \frac{\omega U}{\frac{1}{2} R_s \int_S \left| H_{//} \right|^2 dS}
$$

$$
= \frac{abL(k^2 - k_z^2)^2}{8 R_s \omega \epsilon_0 \left[ \left( \frac{\pi}{a} \right)^2 \frac{bL}{2} + \left( \frac{2\pi}{b} \right)^2 \frac{aL}{2} + \left( \frac{\pi}{a} \right)^2 + \left( \frac{2\pi}{b} \right)^2 \right] \frac{ab}{4}}
$$

(4.92)
5 IP-BPM Design

As discussed in section 3, cavity BPMs have many parameters such as resonant frequency $f_0$, coupling constant $\beta$, and quality factor $Q_{ext}$. Design values of these parameters are summarized in Table 5.1. Also, the whole cavity design is shown in Figure 5.1. In this section, details of IP-BPM fabrication, determination of these parameters and cavity design are discussed.

Since the cavity design was not my work, simulation setups and results are summarized simply. Details are referred in [8, 9, 10] and [11].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>unit</th>
<th>design value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cavity Geometry</td>
<td>mm</td>
<td>61.40 × 48.56 × 6.0</td>
</tr>
<tr>
<td>X Resonant Frequency</td>
<td>GHz</td>
<td>5.712</td>
</tr>
<tr>
<td>Y Resonant Frequency</td>
<td>GHz</td>
<td>6.426</td>
</tr>
<tr>
<td>X Coupling Constant</td>
<td></td>
<td>1.4</td>
</tr>
<tr>
<td>Y Coupling Constant</td>
<td></td>
<td>2.0</td>
</tr>
</tbody>
</table>

Table 5.1: Design parameters of IP-BPM

5.1 Fabrication of IP-BPM

5.1.1 Fabrication Method

The most difficult process in cavity BPM fabrication is connection of the cavity part and the wave guide part. In order to avoid this process, IP-BPM cavity and the wave guide was fabricated from a single block, by end-milling. In this case, inner corners of the cavity and wave guide have to be rounded due to the end-milling. The radius of end-mill used was 4 mm, and therefore IP-BPM was designed considering this R of the inner corners, as is shown in Figure 5.1.
5.1.2 IP-BPM Block

As shown in Figure 5.2, using a single cavity beam position is determined. With two cavities, beam orbit is determined. Moreover with three cavities, position resolution of the cavity can be also determined. Using three cavities, when measured beam jitter is large, it is able to determine if it really originates from beam jitter or from insufficient BPM resolution.

![Figure 5.2: Principle of IP-BPM measurement](image)

Considering beam operation at ATF2, although 3 cavities are needed for calibration and resolution measurements, 2 cavities are enough to monitor beam orbit. Since a line can be always determined from 2 arbitrary points, 2 cavities in a block are convenient. By moving 3rd cavity against the other 2, one can determine a line which runs through all 3 cavities. Therefore 2 + 1 is enough for operation, but instead of fabricating 2 types, we decided to fabricate 2 blocks, 2 cavities in each block.

Required setup is shown in Figure 5.3. Block1, a set of BPM1 and BPM2 is fixed while the other set, Block2 of BPM3 and BPM4, is installed on a precise mover. Calibration of BPM response is possible by moving Block2 against Block1. In this case, BPM3 response of known displacement is compared with extrapolated beam position at BPM3 from monitored beam orbit at BPM1 and BPM2. BPM3 is also used for position resolution measurement.

After calibration, Block1 is enough for beam orbit monitoring. In case of ultra high resolution mode, dynamic range of the BPM is limited in a short range of approximately 5 ~ 10 µm. Since angle jitter is expected to be large at IP due to strong final focus, BPM1 and BPM2 must be located near by to have the beam in dynamic range of both cavities, as shown in Figure 5.4. However, it is not for the case of BPM3, since it is used only for calibration and resolution measurements. At these measurements, not all data is needed but some data must be within dynamic range of all 3 cavities. As a design value, angle jitter at ATF2 IP is maximum 400 µrad, and typically 100 µrad. Assuming 100 µrad angle jitter and IP of ATF2 locating at the center of BPM1 and BPM2, cavity distance d < 100 mm is required. On the other hand, too small d might cause signal coupling between the two cavities. Considering these requirements, d = 76 mm was determined.

![Figure 5.3: Required IP-BPM setup for ATF2](image)

![Figure 5.4: IP-BPM operation at ATF2](image)
5.2 Characteristics of IP-BPM

IP-BPM has 3 main characteristics, compared with other cavity BPMs. Design value of IP-BPM parameters has to be determined to meet these requirements. First, rectangular cavity shape to measure beam position in X direction and Y direction, independently. Second, low angle sensitivity, since large angle jitter due to the strong focus at IP could easily saturate the measurement electronics. Third, ultra high position sensitivity, in order to detect nano-meter beam offset.

IP-BPM cavity shape is designed to be rectangle, while Q-BPMs for ATF2 are cylindrical. This is to measure beam position in both X and Y independently, with a single cavity. Due to different cavity length in X and Y, X di-pole mode $TM_{210}$ and Y di-pole mode $TM_{120}$ have different resonant frequencies. By selecting appropriate frequency at read out of the cavity, it is possible to determine X and Y beam position independently.

At ATF2 IP, maximum 400 $\mu$rad, typically 100 $\mu$rad angle jitter is expected. In order to achieve low angle sensitivity, cavity length in z direction $L$ has to be small, as discussed in section 4.6.1.

Since our target signal level is extremely low, which is approximately -100 dBm for 1 nm offset position signal, ultra high position sensitivity is required. In order to achieve ultra high position sensitivity, improvement of coupling constant $\beta$ is essential. However, too large $\beta$ would easily saturate the detecting electronics and lessen dynamic range.

5.3 Design Parameters

5.3.1 Resonant Frequency $f_0$

Applying (4.52) to (4.17), frequency dependency of di-pole mode signal of a rectangular cavity can be evaluated. Excited $TM_{120}$ mode power at a rectangular cavity is shown in Figure 5.5. Bunch length $\sigma_z = 8$ mm, typical value for ATF beam, is assumed. Also, cavity length in Z direction $L$ is fixed. The abscissas are the cavity length in X and Y respectively, while ordinate is the power, in arbitrary unit. Resonant frequency of such a cavity is determined by (A.52). As seen in Figure 5.5, output power would be maximum at C Band region, approximately 5 $\sim$ 7 GHz.

![Excited di-pole mode power](image)

Figure 5.5: Excited di-pole mode power vs cavity geometry

Since electron beam is synchronized to the accelerating frequency of the ATF DR, 714 MHz, it is convenient to design $f_0$ to be integral multiplication of 714 MHz. This is because considering
multiple bunch operation, signal phase would be equivalent at every bunch, which enables to easily separate bunch to bunch information. Therefore, $f_0$ is designed to be 5.712 GHz (= 714 MHz x 8) and 6.426 GHz (= 714 MHz x 9) for X and Y respectively.

At the beginning, X band operation mode was planned at ATF, which was 2.8 ns interval (357 MHz), 20 bunch. Assuming to suppress phase shift within 10% during this operation, which is 56 ns long, required precision of $f_0$ is 2 MHz. However, X band operation is already given up.

Considering ILC mode operation, which bunch interval is designed to be 154 ns or 308 ns, it is very difficult to tune the phase since required frequency precision would be the order of 100 kHz against C band signal.

Once $f_0$ is determined, rectangular design is determined since $f_0$ for $TM_{210}$ or $TM_{120}$ is mainly determined by cavity size in X and Y direction, $a$ and $b$. From simulation and measurements of test cavities, $a = 61.40$ mm and $b = 48.56$ mm were determined.

### 5.3.2 Cavity Length $L$

Cavity design in X, Y direction is already determined by $f_0$. Therefore, there are 2 free parameters left: cavity length in Z direction $L$, and radius $R_p$ of the beam pipe which contacts the cavity.

As discussed in section 4.6.1 cavity length $L$ has to be shortened in order to reduce angle sensitivity. However as is shown in (4.52), shorter $L$ decreases $R/Q$, which reduces position sensitivity also. To recover position sensitivity, $R_p$ is required to be small, in order to prevent leakage of the field from the cavity.

From simulation results, $L = 6$ mm, $R_p = 3$ mm was determined. However, small beam pipe aperture is not desired at ATF2. Since X position sensitivity is less desired compared to Y, beam pipe radius in X direction was determined to be $R_p = 6$ mm.

### 5.3.3 Coupling Constant $\beta$

As shown in (4.19), position sensitivity is determined by $R/Q$ and $Q_{ext}$. Since cavity shape is completely determined from the first two requirements, $R/Q$ is already determined since it depends on cavity shape only.

In order to increase external coupling, energy loss inside the cavity must be reduced. This fact requires a high $Q_0$ for cavity material. Therefore as shown in (4.72), surface resistance $R_s$ must be small. Considering (4.73), it is achievable by choosing a material with large conductivity $\sigma$. Therefore cavity material was determined to be copper, since its conductivity $5.9 \times 10^7$ (Ω⁻¹m⁻¹) is larger compared to other materials. From cavity material and geometry, $Q_0$ is determined.

Considering (3.6), free parameter left is $Q_{ext}$, and in order to increase coupling $\beta$, low $Q_{ext}$ is required. Low $Q_{ext}$ improves position sensitivity as shown in (4.19), while it also leads to small decay time constant $\tau$. Small $\tau$ easily saturates the electronics and lessen the dynamic range, since stored cavity power is output in very short time. Considering the dynamic range of electronics, $\beta$ was determined to be 1.4 and 2.0 for X and Y, respectively.

The main factor to determine $\beta$ is the position of the slot, due to the coupling between leakage signal from the cavity and propagation mode of the wave guide. For simplification, slot design was assumed to be the same with other ATF2 Q-BPMs, and slot position in relation to the cavity and the wave guide was optimized to maximize $\beta$ by simulation.

Wave guide design was determined first. The geometrical parameters of the wave guide design are shown in Figure 5.6. Considering geometrical limitation at ATF2 beam line, IP-BPM is required to be compact in beam direction, while there is no limitation in transverse direction. Therefore, wave guides for IP-BPM were designed to stretch in transverse direction.

The design of wave guide cross section is determined by cut-off frequency. In order to cut off mono-pole mode of the cavity, which is $f \simeq 3.96$ GHz, cut-off frequency $f_c > 4$ GHz is required. On the other hand, X di-pole mode, $f \simeq 5.71$ GHz, has to be transmitted. Therefore $f_c = 5$ GHz was determined. Cut-off of a wave guide is represented as

\[
f_c = \frac{c}{2\pi} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2},
\]

(5.1)
where \(a, b\) are defined in Figure 5.6, and \(m\) and \(n\) are integers. The lowest frequency mode which can transmit the wave guide is determined from \(m = 1\) and \(n = 1\), and therefore \(a = 30\) mm and \(b = 8\) mm were determined for IP-BPM wave guide.

Wave guide length \(L\) was determined by the following approach. Since wave guide has a finite \(L\), in practice it is considered as a cavity. Thus there are eigenmodes determined by \(L\) and \(a\) or \(b\). If resonant frequency of the wave guide equates the di-pole mode frequency of IP-BPM cavity, contamination between the modes would occur. In this situation, we could not evaluate the desired di-pole mode signal from the output signal. Therefore, \(L\) was determined not to coincide the di-pole mode frequency, and was designed to be 60 mm and 57 mm for X and Y, respectively.

Next, slot position was optimized by simulation. The parameter is the slot position from the edge, \(sp\) (slot position). In case of 5.712 GHz input and 6.426 GHz input from the slot, \(sp\) was optimized to maximize transmittance S parameter. Definition of S parameter is show in section 6.1.

Finally the coaxial antenna position was optimized by simulation. Since there were already antennas designed for ATF2 Q-BPMs, design of the antenna itself was not changed. There are 2 parameters to determine antenna position, which are the distance from wave guide edge \(ap\) (antenna position) and the antenna depth in the wave guide \(al\) (antenna length). If the wave guide to coaxial matching is perfect, reflection S parameter at 5.712 GHz or 6.426 GHz is supposed to be zero. Therefore they were optimized to have no reflection at 5.712 GHz or 6.426 GHz.

The optimized design values of the parameters are summarized in Table 5.2.

![Figure 5.6: Wave guide design parameters](image-url)

<table>
<thead>
<tr>
<th>Mode</th>
<th>slot position (sp) (mm)</th>
<th>antenna length (al) (mm)</th>
<th>antenna position (ap) (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>16.0</td>
<td>5.8</td>
<td>11.4</td>
</tr>
<tr>
<td>Y</td>
<td>17.5</td>
<td>5.8</td>
<td>9.2</td>
</tr>
</tbody>
</table>

Table 5.2: Design parameters of slot and antenna
5.4 Reference Cavity

The rectangular cavity, which di-pole mode is used for monitoring beam position, is called the "sensor cavity". On the other hand, a cylindrical cavity called "reference cavity" is also fabricated. As discussed in section 4, phase detection is critical for separating beam position information from other undesired signals. Also beam charge monitoring is necessary since di-pole signal is also sensitive to beam charge. Reference cavity is used in these purposes.

Mono-pole mode $TM_{010}$ of the reference cavity is used for beam charge monitoring. As discussed in section 4.4, near cavity center mono-pole mode is sensitive only to beam charge and not to beam position. There are two cylindrical cavities fabricated, corresponding to X and Y, respectively. The X reference cavity has a mono-pole mode of 5.712 GHz, while Y has a mono-pole mode of 6.426 GHz. They are designed to match the di-pole mode frequencies of sensor cavities, in order to be used for phase detection. The cylinder design is determined from (A.92). X reference diameter is 21.25 mm and its cavity length is 10 mm. Y reference diameter is 19.07 mm and its cavity length is also 10 mm. The structure of reference cavity is shown in Figure 5.7.

Since excitation of mono-pole mode dominates all the modes, special selective coupler is not necessary. The reference cavity is coupled to a coaxial cable through a small coupling hole on an end plate.

![Reference cavity structure](image)

Figure 5.7: Reference cavity structure, from [5]
6 Basic Tests

6.1 Measurement of basic parameters

As mentioned, 2 blocks of IP-BPM with 2 cavities in each block were fabricated. They are shown in Figure 6.1 and Figure 6.2. After fabricating IP-BPM blocks, their basic parameters such as resonant frequencies, quality factors, and coupling constants were checked.

![IP-BPM block components](image1)

![IP-BPM block appearance](image2)

Figure 6.1: IP-BPM block components

Figure 6.2: IP-BPM block appearance

6.1.1 Measurement Principle

The measurements were carried out by injecting RF signal to the cavity from a port, using a network analyzer. A network analyzer can measure frequency dependency of a considering object by inputting signals of various frequencies to the object through a port, and monitoring output signal intensity.

Basically there are two types of measurement: 1 port measurement and 2 port measurement, as shown in Figure 6.3 and Figure 6.4, respectively. 1 port measurement is carried out for a cavity with 1 port only, such as reference cavities. RF signal is input to the cavity from a network analyzer through the port, and its reflection is measured at the network analyzer. 2 port measurement is carried out for a cavity with more than 2 ports, such as sensor cavities. RF signal is input to the cavity from one port, and its reflection from the same port or its transmission from the other is measured at the network analyzer. Unused ports are terminated by 50 Ω terminators.

To measure resonant frequency and quality factor, 1 port reflection measurement or 2 port transmission measurement is carried out. Network analyzer measures the S-parameter of reflection or transmission, which is the ratio between measured output amplitude and input amplitude. If input power reflects or transmits 100 %, S-parameter would be 1. However in practical, S-parameters would not be 1 even at maximum, due to energy loss at inner wall. Typical measurement results of S-parameters are shown in Figure 6.5. As shown in the figure and following discussion, resonant frequency $f_0$ and loaded quality factor $Q_L$ are obtained from the peak frequency and its width. Resonant frequency is the peak frequency.

At these measurements, $Q_L$ is determined by the following equation:

$$Q_L = \frac{f}{\Delta f}, \tag{6.1}$$

where $\Delta f$ is the frequency width at half power of the stored power at peak frequency, as shown in Figure 6.5. Since S parameters shown in Figure 6.5 represent amplitudes, half power correspond to $\sqrt{\frac{a^2}{2}}$ and $\sqrt{\frac{a^2+b^2}{2}}$ for transmission and reflection, respectively.
Network Analyzer
Cavity
Figure 6.3: 1 Port Measurement

Network Analyzer
Cavity
Figure 6.4: 2 Port Measurement

Figure 6.5: S-parameters of transmission and reflection

(6.1) can be derived easily from (3.10). Since electric field excited at the cavity is proportional to the square root of stored energy $U$, it is written as

$$E(t) = E_0 \exp \left(-\frac{\omega_0 t}{2Q_L}\right) e^{i\omega_0 t}, \quad (6.2)$$

where $\omega_0$ is the resonant angular frequency. Noticing that the electric field is excited at $t = 0$, the frequency spectrum is obtained by Fourier Transform of (6.2),

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty E(t)e^{-i\omega t}dt$$

$$= \frac{E_0}{\sqrt{2\pi}} \int_0^\infty \exp \left[-\frac{\omega_0}{2Q_L} - i(\omega - \omega_0)\right] tdt$$

$$= \frac{E_0}{\sqrt{2\pi}} \frac{\omega_0}{\frac{\omega_0}{2Q_L} + i(\omega - \omega_0)}, \quad (6.3)$$

Since the stored energy of the cavity is proportional to the square of this amplitude,

$$U(\omega) \propto |E(\omega)|^2 \propto \frac{1}{\left(\frac{\omega_0}{2Q_L}\right)^2 + (\omega - \omega_0)^2}, \quad (6.4)$$
is obtained. This becomes the maximum at $\omega = \omega_0$, and the value is

$$U = U_{MAX} = \left(\frac{2QL}{\omega_0}\right)^2. \quad (6.5)$$

Considering the half of maximum energy, the condition would be

$$U(\omega) = \frac{1}{2} U_{MAX}$$
$$\omega = \omega_0 \pm \frac{\omega_0}{2QL}, \quad (6.6)$$

and FWHM (Full Width at Half Maximum) $\Delta\omega$ is represented as

$$\Delta\omega = \frac{\omega_0}{Q_L}. \quad (6.7)$$

This leads to (6.1), by

$$\frac{f_0}{\Delta f} = \frac{\omega_0}{\Delta\omega} = Q_L. \quad (6.8)$$

The expression of coupling constant $\beta$ varies due to the way of measurement. 2 port measurement case is discussed first. Cavity BPMs used at ATF2 is designed to have good symmetry, thus 2 ports are installed symmetrically for each dipole mode. Since signals from the 2 ports are combined after output, $\beta$ of this case represents the coupling of the sum of the 2 ports. As shown in Figure 6.4 let’s assume that RF of amplitude $a$ is input from a port, and the reflection or the transmission is measured by a network analyzer. When input RF signal frequency differs from the resonant frequency of the cavity, signal is reflected and cannot enter the cavity. However, when input frequency meets the resonant frequency, it can be stored in the cavity. When the leakage amplitude is $b$ ($b < a$), the amplitude measured at input side is $a - b$, since phase of the reflection is flipped. Considering symmetry of the 2 ports, leakage at the other side is $b$. Thus the amplitude reflectance $S_{11}$, transmittance $S_{21}$ are written as

$$S_{11} = \frac{a - b}{a} \quad (6.9)$$
$$S_{21} = \frac{b}{a}. \quad (6.10)$$

$S_{11}$ and $S_{21}$ are the S parameters of reflection and transmission, respectively.

Considering a steady state with input $a$ and leakage $b$, there is no way in principle to discriminate $b$ from $a$ at the input side. In this case the reflection is considered to be a single RF of $a - b$, and not the superposition of $a$ and $b$. Therefore reflecting energy is $(a - b)^2$. Also noticing that the input energy is $a^2$ and transmitting energy is $b^2$, energy loss at the cavity wall $P_{wall}$ is derived from the energy conservation law as

$$P_{wall} = a^2 - (a - b)^2 - b^2 = 2b(a - b). \quad (6.11)$$

On the other hand, sum of the output energy just after the input $a$ is turned off is calculated as

$$P_{out} = b^2 + b^2 = 2b^2. \quad (6.12)$$

Since $P_{wall}$ and $P_{out}$ are continuous at the instant the input is turned off, $P_{wall}$ and $P_{out}$ are comparable. Therefore from the definition of $\beta$ (3.6),

$$\beta = \frac{P_{out}}{P_{wall}} = \frac{b}{a - b}. \quad (6.13)$$
is obtained. Applying \((6.9)\), it is written as
\[
\beta = \frac{1 - \frac{a-b}{a}}{\frac{a-b}{a}} = \frac{1 - S_{11}}{S_{11}},
\]
using S parameter \(S_{11}\). Applying \((6.10)\), it is also written as
\[
\beta = \frac{b}{1 - \frac{b}{a}} = \frac{S_{21}}{1 - S_{21}},
\]
using S parameter \(S_{21}\).

Measurement method is the same in case of 1 port measurement also. Let’s assume that RF of amplitude \(a\) is input from a port, and the reflection is measured by a network analyzer. When input RF signal frequency differs from the resonant frequency of the cavity, the signal is reflected and cannot enter the cavity. However, when the input frequency meets the resonant frequency, it can be stored in the cavity. When the leakage amplitude is \(b\) \((b < a)\), amplitude measured as reflection is \(a - b\). Thus the amplitude reflectance \(S_{11}\) is written as
\[
S_{11} = \frac{a-b}{a}.
\]
Noticing that the input energy is \(a^2\) and reflecting energy is \((a - b)^2\), energy loss at the cavity wall \(P_{\text{wall}}\) is derived from the energy conservation law,
\[
P_{\text{wall}} = a^2 - (a - b)^2 = b(2a - b).
\]
On the other hand, the output energy is
\[
P_{\text{out}} = b^2.
\]
From the definition of \(\beta\) \((3.6)\) and \((6.16)\),
\[
\beta = \frac{P_{\text{out}}}{P_{\text{wall}}} = \frac{b}{2a - b} = \frac{1 - \frac{a-b}{a}}{1 + \frac{a-b}{a}} = \frac{1 - S_{11}}{1 + S_{11}}
\]
is obtained.

From \(Q_L\) and \(\beta\), one can calculate \(Q_0\) and \(Q_{\text{ext}}\) using \((3.7)\) and \((3.8)\), respectively. \(Q_{\text{ext}}\) is essential to check the cavity design, since it depends only on cavity shape and not on material or surface condition of the cavity.

### 6.1.2 Prediction from theory

Resonant frequency \(f_0\) and quality factor \(Q_0\) can be calculated analytically in case of approximate geometry of IP-BPM, which is a rectangle of \(61.40 \times 48.56 \times 6\) mm. Frequency \(f_0\) can be calculated by applying TM\(_{210}\) or TM\(_{120}\) to \((A.52)\). Also, \(Q_0\) can be calculated from \((1.92)\). The applied parameters for our cavity material, copper, are conductivity \(\sigma = 5.9 \times 10^7\) \((1/\Omega\text{m})\) and magnetic permeability \(\mu = \mu_0 = 4\pi \times 10^{-7}\). Their results are summarized in Table 6.1.

### 6.1.3 HFSS Simulation

A simulation was made to evaluate basic parameters using HFSS ver9, an electromagnetic field calculator. HFSS can calculate electromagnetic field and its time evolution excited in a considering object. There are typically two simulation modes, the Driven Mode and the Eigen Mode. Driven Mode is used for calculating S-parameters of transmittance or reflectance, between user defined wave ports. Eigen Mode is used to calculate eigen-modes of the considering geometry. Also, values
such as stored energy and field strength can be calculated using the Field Calculator. Various materials and boundary conditions can be applied to simulation objects.

In case of calculating quality factors, Driven Mode was applied. The geometry used for simulation is shown in Figure 6.6. It is consisting of a cavity part, 50 mm beam pipe at each side of the cavity, slot, waveguide, and coaxial antennas. The cavity design is exactly the same with the actual IP-BPM.

Figure 6.6: Simulation geometry for Q measurement

Boundary condition of the geometry surface was set to be copper, same as actual IP-BPM. Each antenna was set as wave ports, and RF signal was input from those ports. HFSS calculates the S parameter $S_{21}$ and $S_{11}$, which are defined in (6.10) and (6.9) respectively. This simulation setup is same as the transmission or reflection measurement using a network analyzer. Thus, the transmittance $S_{21}$ is expected to be a Lorentzian as shown in (6.4), and $S_{21}$ can be fitted by function

$$S_{21}(\omega) = C \times \frac{\omega_0^2}{\sqrt{\left(\frac{\omega_0}{2Q_L}\right)^2 + (\omega - \omega_0)^2}},$$  \hspace{1cm} (6.20)

where $C \ (0 < C < 1)$ is a constant, and $\omega_0$ is the resonant frequency. Due to energy loss at inner wall $P_{\text{wall}}$, $C$ is smaller than 1. The energy loss occurs since copper has a finite conductivity. When frequency $\omega$ equate the resonant frequency $\omega_0$, S Parameter takes the maximum, which is

$$S_{21}^0 = C.$$  \hspace{1cm} (6.21)

From (3.7), (3.8) and (6.15), $Q_0$ and $Q_{\text{ext}}$ can also be calculated as

$$Q_0 = (1 + \beta)Q_L = \frac{1}{1 - S_{21}^0}Q_L$$

<table>
<thead>
<tr>
<th>Mode</th>
<th>$f_0$ (GHz)</th>
<th>$Q_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>X di-pole</td>
<td>5.7805</td>
<td>5852</td>
</tr>
<tr>
<td>Y di-pole</td>
<td>6.6434</td>
<td>6312</td>
</tr>
</tbody>
</table>

Table 6.1: Calculated parameters of IP-BPM
\[ Q_{\text{ext}} = \frac{1 + \beta}{\beta} Q_L = \frac{1}{S_{21}} Q_L. \] (6.22)

The simulated S Parameter \( S_{21} \) for X, Y di-pole mode is shown in Figure 6.7 and Figure 6.8 respectively. Also, the fitted resonant frequency \( f_0 \), \( Q_{\text{ext}} \) and other parameters are summarized in Table 6.2. Since geometry is different, it cannot be simply compared with Table 6.1, however it is basically consistent with the calculated values. As shown in the table, design values of external coupling, \( \beta = 1.4 \) for X and \( \beta = 2.0 \) for Y, are achieved.

![Figure 6.7: S parameter for X-port Transmission](image)

![Figure 6.8: S parameter for Y-port Transmission](image)

| Mode     | \( f_0 \) (GHz) | \( \beta \) | \( Q_L \) | \( Q_0 \) | \( Q_{\text{ext}} \) | \( \tau \) (ns) |
|----------|----------------|------------|-----------|-----------|*********************|**********|
| X di-pole| 5.7086         | 1.578      | 2070      | 5337      | 3382              | 58          |
| Y di-pole| 6.4336         | 3.154      | 1207      | 5015      | 1590              | 30          |

Table 6.2: Simulated parameters of IP-BPM

### 6.1.4 Results

The measurement results are summarized in Table 6.3. As discussed in advance, \( f_0 \), \( \beta \) and \( Q_L \) are measured directly by a network analyzer, and \( Q_0 \), \( Q_{\text{ext}} \) are calculated from those results. Also, decay time constant of the cavity \( \tau \) is calculated using (3.11).

The measurement was carried out under air pressure condition. Please note that compared with air pressure condition, \( f_0 \) increases at vacuum condition. This is because at vacuum condition effect of refraction factor of the atmosphere disappears and effective length of the cavity reduces. In case of our IP-BPM, frequency increase was about approximately 2 MHz. Results are consistent with the simulation.

### 6.2 Measurement of X-Y isolation

Since IP-BPM cavity is a rectangular cavity, resonant frequencies in X and Y differ. Therefore if cavity is perfect, there would be no coupling between X di-pole mode signal and Y di-pole mode signal.

X-Y isolation can also be measured by a network analyzer. Resonant frequency signal is input from a port, and output from the opposite port and transverse port are monitored as shown in Figure 6.9. In ideal condition, the signal would transmit to the opposite port and does not
transmit to the transverse port. By measuring the S parameter of both transmission, X-Y isolation is defined as

$$20 \log_{10} \left( \frac{S_{\text{transverse}}}{S_{\text{opposite}}} \right) \quad (dB).$$  \hspace{1cm} (6.23)

As a result of our measurement, X-Y contamination was not observed, which was under noise level, if existed. X-Y isolation was confirmed to be under -50 dB, which implies that X-Y power coupling, $P_x/P_y$ or $P_y/P_x$, is lower than $10^{-5}$. Therefore IP-BPM was proved to be able to measure X and Y signal completely independently.

![Figure 6.9: schematics of X-Y isolation measurement](image)

### 6.3 Measurement of $R/Q$

#### 6.3.1 Estimation from theory

Applying the IP-BPM cavity property $a = 61.40 \text{ mm}$, $b = 48.56 \text{ mm}$, $L = 6.0 \text{ mm}$ and $\omega = 2\pi \times 6.426 \text{ (GHz)}$ to \[4.52\],

$$\frac{R}{Q}(y) \simeq 0.69y^2 \quad \hspace{1cm} (6.24)$$
is obtained. Also for X di-pole mode,
\[ \frac{R}{Q}(x) \simeq 0.50x^2 \]  
(6.25)
is obtained.

### 6.3.2 Simulation by HFSS

Simulation was made by using HFSS ver9. The geometry used for simulation is shown in Figure 6.10. It is consist of cavity part and 50 mm beam pipe at each side, and the cavity design is exactly the same with actual IP-BPM cavity. Slots and waveguides are not considered in this simulation. Simulation was made by "Eigen Mode" of HFSS.

![Simulation geometry for R/Q measurement](image)

**Figure 6.10:** Simulation geometry for R/Q measurement

\( R/Q \) was calculated using "Field Calculator" of HFSS, by applying the definition of \( R/Q \), (3.13). The stored energy \( U \) was calculated by

\[ U = \frac{1}{2} \int_V (\mu |H|^2 + \epsilon |E|^2) dV, \]  
(6.26)

where \( V \) is the cavity volume. Also, \( E_z \) was integrated along the z direction with a fixed X or Y offset. Since HFSS simulation does not consider phase transition, transit time factor \( T \) has to be multiplied to the simulated \( R/Q \). At this point, effective cavity length \( L_{\text{eff}} \) was defined as the length where the field strength \( E_z \) is over \( 1/e \) of the field strength at the cavity center. \( L_{\text{eff}} \) was also determined by simulation, as shown in Figure 6.11. The origin of abscissa represents the cavity center, where amplitude is maximum.

At first, simulation spec was checked by running simulation with a simple geometry, and compared with analytical solutions, (6.24) and (6.25). The geometry was 61.40 mm × 48.56 mm × 6 mm rectangular. Resonant frequency \( f_0 \) and the proportionality coefficient of \( R/Q \) vs square of offset, \( \alpha \) are compared. The results are summarized in Table 6.4. As shown in the table, resonant frequency is consistent, and \( \alpha \) is also consistent within about 10 %.

The simulation results are shown in Figure 6.12. As seen in the figure, \( R/Q \) has a square correlation to the offset, which is consistent with (6.24) and (6.25). The simulation geometry is not an exact rectangular due to finite \( R \) at inner corners, and also there are effects from beam pipes. Difference between the simulation result and the calculated (6.24) and (6.25) is thought to be due to this geometry difference.
<table>
<thead>
<tr>
<th>Mode</th>
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<th>( f ) (GHz)</th>
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<td>X di-pole</td>
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<td></td>
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Table 6.4: Results of HFSS simulation check using simple geometry

![Figure 6.11: \( E_z \) distribution in longitudinal direction, from [12]](image)

![Figure 6.12: \( R/Q \) Simulation by HFSS](image)

### 6.3.3 Measurement Principle

At \( R/Q \) measurement, bead perturbation method is often applied. \( R/Q \) is determined from the shift of resonant frequency, by giving a small perturbation to the electric field of the cavity, such as entering a small conductive sphere into the cavity. Assuming that a small volume \( \Delta V \) is inserted to the cavity, resonant frequency \( \omega \) shifts from the original frequency \( \omega_0 \). As discussed in Appendix A.4 from the Slater’s Perturbation Theory [13] it is given as

\[
\omega^2 = \omega_0^2 \left[ 1 + \frac{\int_{\Delta V} (\mu |H|^2 - \epsilon |E|^2) dV}{2U} \right]
\]

\[
\omega^2 - \omega_0^2 = \frac{\int_{\Delta V} (\mu |H|^2 - \epsilon |E|^2) dV}{2U},
\]

(6.27)

where \( \mathbf{H}, \mathbf{E}, \mu, \epsilon \) are the magnetic field, electric field, permeability, and permittivity, respectively. Defining the frequency shift \( \Delta \omega \equiv |\omega - \omega_0| \ll \omega_0 \) and neglecting the square term of \( \Delta \omega/\omega_0 \),

\[
\frac{\omega^2 - \omega_0^2}{\omega_0^2} = \left( \frac{\Delta \omega}{\omega_0} \right)^2 + 2 \frac{\Delta \omega}{\omega_0} \approx \frac{\int_{\Delta V} (\mu |H|^2 - \epsilon |E|^2) dV}{2U}
\]

\[
\frac{\Delta \omega}{\omega_0} = \frac{\int_{\Delta V} (\mu |H|^2 - \epsilon |E|^2) dV}{4U},
\]

(6.28)

is obtained. Note that \( \omega_0 = 2\pi f_0 \), \( \Delta \omega = 2\pi \Delta f \) are used.

At actual measurement, ceramic (alumina) bead was used as a small perturbation. Ceramic has a large relative permittivity \( \epsilon_r \), while its relative magnetic permeability is \( \mu_r = 1 \), effectively same as vacuum. Therefore ceramic is sensitive to electric field and not to magnetic field, and
distribution of the electric field inside the cavity can be determined. Assuming that the electric field being constant near by bead, (6.28) is written as

$$\Delta f = -\frac{\int_{\Delta V} |E|^2 dV}{4U} = -\frac{\epsilon |E|^2 \Delta V}{4U}.$$  
$$|E| = 2\sqrt{-\frac{U \Delta f}{\epsilon \Delta V}}.$$  

(6.29)

Applying (6.29) to (3.13),

$$\frac{R}{Q} = \frac{\int |E| ds}{\omega_0 U} = \frac{\int 2 \sqrt{-\frac{U \Delta f}{\epsilon \Delta V}} ds}{\omega_0 U} = \frac{2 \int \sqrt{\Delta f} |ds|^2}{\pi f_0^2 \epsilon \Delta V}.$$  

(6.30)

is obtained. It is shown that $R/Q$ is determined by integrating the frequency shift $\Delta f$ while inserting a bead along the beam orbit.

### 6.3.4 Measurement Scheme

The scheme of bead perturbation measurement is shown in Figure 6.13. To carry out the measurement, we used a ceramic bead attached to a nylon wire. This wire was inserted into the cavity through a fringe, and was modulated with constant step in z direction (beam direction) using a motor drive unit. During the measurement, RF signal was input from the port and its reflection was monitored using a network analyzer. Thus one can observe the resonant frequency shift while modulating the bead inside the cavity. The bead used was consist of 93% alumina, and its volume was $\Delta V = 7.80 \text{ mm}^3$, relative permittivity was $\epsilon_r = 8.5$.

Bead scan was performed for the cavity center, on X axis with various Y offset, and on Y axis with various X offset. First the wire was aligned at the cavity center by eye. Then, bead was slid into the cavity, while frequency of the cavity was monitored by the network analyzer. Scanning the bead to z direction, z center of the cavity was determined by searching the minimum reflection point. Next, bead was slid to X, Y direction, and center of X and Y direction were determined by searching the maximum resonant frequency.

By scanning the cavity with various Y offsets, $R/Q$ of the offset is calculated using (6.30), and therefore Y dependence of $R/Q$ is obtained. From its symmetry, precise Y center is determined since $R/Q$ would be minimum at Y center. Same method is applied to determine precise X center also.

Once X, Y center are determined, again the cavity was scanned by the bead with various offsets on Y axis and X axis. The scan was taken for 80 mm along z direction, including cavity length of 6 mm. To check its reproducibility the measurement was made twice for each offset measurement.

### 6.3.5 Results

Resonant frequency shift of the cavity is shown in Figure 6.14 and Figure 6.15 for X offset and Y offset, respectively. Abscissa is the z position, and ordinate is the resonant frequency of the cavity. X offset scan was taken with 0.8 mm step size to $\pm 2.4$ mm, while Y offset scan was taken with 0.5 mm step size to $\pm 1.0$ mm. Please note that the beam pipe diameter attached to the cavity is 3.0 mm for X and 1.5 mm for Y.

$R/Q$ calculated from (6.30) for X and Y are shown in Figure 6.16 and Figure 6.17 respectively. As seen in these figures, $R/Q$ has a square correlation to the offset, which is consistent with (6.24) and (6.25).

Considering di-pole modes, ideally there is no electric field on central axis, and therefore $R/Q$ is zero. In bead perturbation measurement, $R/Q$ is non-zero even on the central axis due to the finite volume of the bead itself. Although scanning on the central axis, bead would feel the electrical field of its volume (or its permittivity).
Basic parameters of reference cavities were also checked by 1 port reflection measurement, under vacuum condition. Frequency tuner ports were prepared on reference cavities, in order to tune the frequency by pushing or pulling the cavity walls. By pushing the cavity wall, effective length of the cavity decreases and the resonant frequency increases, while it is the opposite for pulling the cavity wall. The schematics of frequency tuning are shown in Figure 6.18.

Table 6.4 shows the measured parameters of the mono-pole mode of each reference cavity, before the frequency tuning. Simulation and predicted values are also shown. Predicted values are calculated by assuming the cavity geometry as a cylinder and applying the geometry to (A.92) and (4.83). Although external coupling differs between measured values and simulation values, it is not critical since mono-pole mode dominates other modes and is strong enough, as shown in Figure 4.4. This is due to large $R/Q$, which is maximum at cavity center.

Using tuner ports, mono-pole mode frequencies of reference cavities were adjusted to match with the di-pole mode frequencies of the sensor cavities.
Figure 6.16: Measured R/Q of X di-pole mode

Figure 6.17: Measured R/Q of Y di-pole mode

Figure 6.18: Schematics of reference cavity frequency tuning

<table>
<thead>
<tr>
<th></th>
<th>( f_0 ) (GHz)</th>
<th>( \beta )</th>
<th>( Q_L )</th>
<th>( Q_0 )</th>
<th>( Q_{ext} )</th>
<th>( \tau ) (ns)</th>
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</table>

Table 6.5: Measured basic parameters of reference cavities
7 Detecting Scheme

In case of position sensitivity or angle sensitivity tests, which are discussed in section 9, we could simply apply diodes to detect signal amplitude. However, it is not enough to monitor beam position at a cavity, since from the amplitude we can determine only absolute value of the beam offset and not its direction from the cavity center. Therefore in order to detect beam position, phase detection is going to be critical. Also by detecting signal phase, it is possible to detect beam position information independently from other noise components, such as angle jitters or bunch tilts, as discussed in section 4.6. These phase differences are shown in Figure 7.1.

In this section, method of phase detection, thermal noise of electronics and the whole scheme of beam position detection are discussed in detail.

![Figure 7.1: Phase difference of sensor cavity signals](image)

7.1 Phase Detection

The most essential module for phase detection is a mixer. A mixer is a multiplier device, used to down convert RF signal by detecting frequency difference of two input sin waves. Assume that two sin waves \( Y_1(t) \) and \( Y_2(t) \) of frequency \( \omega_1 \) and \( \omega_2 \) respectively, are input to a mixer. The two sin waves are represented as below:

\[
Y_1(t) = A \sin(\omega_1 t + \phi_1) \\
Y_2(t) = B \sin(\omega_2 t + \phi_2),
\]

where \( \phi_1, \phi_2 \) are the initial phase of \( Y_1(t) \) and \( Y_2(t) \), respectively. Output of the mixer would be a simple multiplication of the two inputs, which is

\[
Y_1(t) \times Y_2(t) = AB \sin(\omega_1 t + \phi_1) \sin(\omega_2 t + \phi_2) \\
= \frac{AB}{2} \cos((\omega_1 - \omega_2)t + (\phi_1 - \phi_2)) - \cos((\omega_1 + \omega_2)t + (\phi_1 + \phi_2)).
\]

As one can see, a mixer outputs the difference frequency and sum frequency of the input. In order to acquire the difference frequency, one can put a low pass filter after the mixer. Thus the output would be

\[
\frac{AB}{2} \cos((\omega_1 - \omega_2)t + (\phi_1 - \phi_2)).
\]
DETECTING SCHEME

7.1 DETECTING SCHEME

7.1.1 Analog Detection

In case of single bunch operation, analog phase detection is carried out. The detecting circuit is shown in Figure 7.3. In Figure 7.3, upstream of the beam line is shown in the upper side. Please note that there are similar scheme for every X and Y of all 3 sensor cavities, although only Y signal of one sensor cavity is shown in the figure.

In this section, details of each component are described from the upstream, by taking Y signal as an example.

From now on, calculations are focused on signal phase only, and signals are normalized by their amplitudes. For convenience, frequency of DR accelerating RF, 714 MHz, is defined as \( \omega_0 \). Signal from sensor cavity or reference cavity are 6.426 GHz for Y, which is 9\( \omega_0 \), 5.712 GHz LO (Local Oscillator) signal is used in this case. Generally analog detection is preferred, since digitalizing and analyzing require longer time to acquire phase information. In order to give fast feedback to beam position using IP-BPM signal, this dead time is undesirable. Also, precision of phase detection is considered to decrease in case of digital detection, since it is limited by ADC bit number.

As described in section 4.6, noise components are \( \pi/2 \) different in phase compared to position signals. Thus in order to detect position signals precisely, signal phase must be detected, which require a beam synchronized phase origin. Reference cavity is used in this purpose, as shown in Figure 7.2. The phase difference between position signal and reference signal corresponds to the distance between the two cavities, which is always constant.

There are basically two ways to detect signal phase, which are analog detection and digital detection. In case of analog detection, signal phase is detected with a beam synchronized reference signal, and output of the detector would be the phase difference as shown in (7.4). In case of digital detection, signal phase is detected in analysis after digitalizing. Unlocked LO (Local Oscillator) signal is used in this case. Generally analog detection is preferred, since digitalizing and analyzing require longer time to acquire phase information. In order to give fast feedback to beam position using IP-BPM signal, this dead time is undesirable. Also, precision of phase detection is considered to decrease in case of digital detection, since it is limited by ADC bit number.

As described in section 11, digital detection was carried out under multiple bunch operation. This was because the sensor and reference signals are summed up, and phase information of every bunch is being mixed. Thus signal phase of a single bunch could not be simply detected by analog detection, and it is necessary to separate bunch to bunch information by analysis.

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Figure 7.2: Principle of phase detection using a reference cavity
Oscillator) signal is generated by octupling DR RF, which is $8\omega_0$. This LO used for first down conversion is locked to DR 714 MHz. Therefore, sensor signal, reference signal and LO signal are written as

$$Y_{\text{sen}} = \sin(9\omega_0 t + \phi_{\text{sen}})$$
$$Y_{\text{ref}} = \sin(9\omega_0 t + \phi_{\text{ref}})$$
$$Y_{\text{LO}} = \sin(8\omega_0 t + 8\phi_{\text{LO}})$$

where $\phi_{\text{sen}}$, $\phi_{\text{ref}}$ and $\phi_{\text{LO}}$ are their initial phase. Sensor and reference signals are mixed with the LO signal and their differential frequency is output from the first down converter, which are

$$Y_{\text{sen-LO}} = \sin(\omega_0 t + (\phi_{\text{sen}} - 8\phi_{\text{LO}}))$$
$$Y_{\text{ref-LO}} = \sin(\omega_0 t + (\phi_{\text{ref}} - 8\phi_{\text{LO}}))$$

At the second down converter, signal phase is detected by mixing sensor signal and reference signal, such as

$$Y_{\text{sen-ref}} = \sin((\omega_0 t + \phi_{\text{sen}} - 8\phi_{\text{LO}}) - (\omega_0 t + \phi_{\text{ref}} - 8\phi_{\text{LO}}))$$
$$= \sin(\phi_{\text{sen}} - \phi_{\text{ref}})$$

In order to acquire phase information, two orthogonal outputs, I and Q are output from phase detector, which are represented as

$$I = \cos(\phi_{\text{sen}} - \phi_{\text{ref}})$$
$$Q = \sin(\phi_{\text{sen}} - \phi_{\text{ref}})$$

Using I and Q, amplitude $A_{\text{sen}}$ and phase of sensor signal $\phi_{\text{sen}} - \phi_{\text{ref}}$ can be obtained from a simple calculation,

$$A_{\text{sen}} = \sqrt{I^2 + Q^2}$$
$$\phi_{\text{sen}} - \phi_{\text{ref}} = \tan^{-1} \left( \frac{Q}{I} \right)$$

As you can see, key point of the phase detection is down converting the sensor signal and reference signal with a common LO, to maintain phase relation between the two signals.

From now on, each components shown in Figure 7.3 are described from upstream.

**Combiner** 6.426 GHz RF signals read out from a pair of ports in opposed position of the sensor cavity first enter the combiner. The combiner outputs their differential signal. The 2 dipole mode signals are opposite in phase, so actually they are doubled. Also by taking the difference, noise signals which are in the modes other than the dipole mode are expected to cancel out.

**Variable Attenuator** Variable Attenuators are used to reduce position sensitivity of the sensor signal, and cover every 10 dB from 10 dB to 70 dB. The purpose of reducing position sensitivity is to enlarge dynamic range of the detecting electronics for calibration. They are consist of a simple relay circuit, and remote control is possible from the "eel’s bedroom", which is the control room of the ATF extraction line. Schematics of the variable attenuator are shown in Figure 7.4.

**Down Converter** After the variable attenuator, RF signal enters the first down converter. Since RF signals decay very quickly, it is critical to install the down converter system as near to the cavity as possible and reduce the frequency. 6.426 GHz sensor or reference signal is mixed with 5.712 GHz LO (Local Oscillator) signal, and the differential part, 714 MHz signal is output. Schematics of the first down converter are shown in Figure 7.5.
Figure 7.3: Block Diagram of Analog Detection
Figure 7.4: Variable Attenuator

Figure 7.5: Block Diagram of Down Converter
Locked LO Generator 5.712 GHz used for first down conversion is generated from locked LO generator. This generator outputs 5.712 GHz signal locked to external input, which is DR accelerating 714 MHz signal. It generates octuple signal of input 714 MHz using non-linearity of the electronics. When amplifier saturates and its output signal is distorted, higher harmonics of the fundamental frequency occur, including octuple frequency. The octuple signal is selected by a band pass filter, and is output from the generator.

Limiter Detector Reference signal enters the limiter detector before used for phase detection. Limiter detector has 2 main functions. First is to output 714 MHz signal of constant amplitude used for phase detection. Second is to monitor the amplitude of reference signal, in order to monitor beam charge. Reference signal is split into 2 and while one enters a limiter amplifier, the other is detected by a diode. Schematics of the limiter detector are shown in Figure 7.6.

Phase Detector The sensor and reference 714 MHz signals are transported a long way to where we call the ”eel’s bedroom”, and enters the phase detector. At the phase detector, 714 MHz sensor signal is mixed with 714 MHz reference signal generated from the limiter detector. As discussed in advance, signal phase is detected in this method. The 714 MHz reference signal is split into 2 and one is $\pi/2$ shifted in phase against the other. The 2 reference signals are mixed with sensor signals, which are also split into 2. By this method, we can obtain 2 signals, I and Q, different for $\pi/2$ in phase. We used a phase shifter of the detector to shift the phase of reference signal, which details are discussed later. Schematics of the phase detector are shown in Figure 7.7. Actually there are 4 outputs from the phase detector to acquire phase information, which are 0, 45, 90 and 135 degrees. However as shown in (7.12), a set of 2 signals different for 90 degrees is enough to determine signal phase. Also from limitation of ADC channels, we used only 0 and 90, which we call I and Q, respectively.

7.1.2 Digital Detection

The scheme of digital phase detection is shown in Figure 7.8. Components used for phase detection is the same as analog detection, so their details are omitted.

In this case, both sensor signals and reference signal are detected by a common unlocked LO 714 MHz signal at second down conversion. Using common LO, phase relation of the two is maintained. It is unlocked to the beam, since it is generated by a signal generator. Thus detected signal at the phase detector would be,

$$
I_{sen} = A_{sen}\sin(\phi_{sen} - \phi_{LO} - \phi_{unlockedLO})
$$

$$
Q_{sen} = A_{sen}\cos(\phi_{sen} - \phi_{LO} - \phi_{unlockedLO})
$$

$$
I_{ref} = A_{ref}\sin(\phi_{ref} - \phi_{LO} - \phi_{unlockedLO})
$$

$$
Q_{ref} = A_{ref}\cos(\phi_{ref} - \phi_{LO} - \phi_{unlockedLO}),
$$

(7.13)
where $\phi_{LO}$ is the initial phase of beam locked LO used for first down conversion, and $\phi_{unlockedLO}$ is the initial phase of unlocked LO used for second down conversion.

In case of analog detection, detected phase $\phi_{sen} - \phi_{ref}$ was constant. However in case of digital detection, detected phase shifts arbitrarily, due to unlocked component $\phi_{unlockedLO}$. Therefore by plotting I and Q in phase surface, it would form a circle if amplitude is stable, as shown in Figure 7.9. From the I-Q plot, amplitude $A$ and relative phase $\phi$ are analytically obtained, by calculating

$$A_{sen(ref)} = \sqrt{(I_{sen(ref)} - I_0)^2 + (Q_{sen(ref)} - Q_0)^2},$$

$$\phi_{sen(ref)} - \phi_{LO} = \tan^{-1} \left( \frac{Q_{sen(ref)} - Q_0}{I_{sen(ref)} - I_0} \right) + \frac{\pi}{2} \left( 1 - \frac{I_{sen(ref)} - I_0}{|I_{sen(ref)} - I_0|} \right),$$

where $I_0$ and $Q_0$ are the center in phase space, determined by measuring pedestal of the detector. Also, $\phi_{LO} + \phi_{unlockedLO}$ was redefined as $\phi_{LO}$.

Finally, signal phase is calculated by using results of (7.14), such as

$$\phi_{sen} - \phi_{ref} = (\phi_{sen} - \phi_{LO}) - (\phi_{ref} - \phi_{LO}).$$

### 7.1.3 I-Q Tuning

Applying these phase detection methods, position signal is separated from other noise components.

In case of analog detection, it is possible to tune position signal to $I$ and the other noise components to $Q$, since detected phase is locked. The I-Q tuning is done by optimizing the signal phase using a phase shifter of the phase detector. $I$ and Q were monitored by an oscilloscope, while sweeping the beam. Signal phase was tuned to make the I signal have the maximum sensitivity to position displacement, while Q signal have none. This procedure was at first applied at 30 dB attenuation, and was repeated in lower attenuations, such as 20 dB and 10 dB. The condition of I-Q tuning is shown in Figure 7.10. As shown in the figure, I response is sensitive to beam displacement, while Q response is almost constant while sweeping the beam. The I signal flips when beam passes with offset in opposite direction, since signal phase shifts for $\pi$.

In case of digital detection, I-Q tuning cannot be done since the LO used for phase detection is not locked to the beam timing. Thus detected phase is not fixed and would be random. In this case, acquirement of beam position information is done by analysis.
Figure 7.8: Block Diagram of Digital Detection
7 DETECTING SCHEME

Figure 7.9: Amplitude and phase detection by analysis

Figure 7.10: I-Q tuning using a phase shifter at analog detection

7.2 Spec of electronics

Position sensitivity, or resolution of IP-BPM is principally limited by thermal noise of the electronics. In order to detect very small signal of nano-meter beam offset, noise of the electronics is required to be extremely low. In this section, it is shown that IP-BPM has a sufficient accuracy to detect 1 nm position signal, compared with the noise level of detecting system.

7.2.1 Noise Figure

The spec of electronics is evaluated by noise figure (NF), which is the ratio of S/N at output ($S_{out}/N_{out}$) and input ($S_{in}/N_{in}$) of the considering system. $S_{out}/N_{out}$ is worse than $S_{in}/N_{in}$, due to the noise of electronics. NF is defined as below:

$$NF = 10\log_{10} \frac{S_{in}/N_{in}}{S_{out}/N_{out}} \equiv 10\log_{10} F.$$ (7.16)

Unit of NF is dB, and if the electronics is ideal, which is free of noise, NF would be 0 dB.

Let’s assume that $n$ pieces of amplifiers are aligned in series, where NF and gain of each module is $F_1, F_2, \cdots, F_n$ and $G_1, G_2, \cdots, G_n$. According to Friss’s formula, NF of whole electronics scheme

$$NF = 10\log_{10} \frac{S_{in}/N_{in}}{S_{out}/N_{out}} \equiv 10\log_{10} F.$$ (7.16)_1
Parameter | min | typ | MAX | unit
--- | --- | --- | --- | ---
Frequency | 4 | 8 | GHz |
Gain | 29 | 30 | dB |
Power@1dB Compression Point | 15 | 16 | dBm |
NF(Noise Figure) | 1.7 | 2.2 | dB |

Table 7.1: NBL00419 spec sheet, from [14]

is obtained as

\[ F_{\text{total}} = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1G_2} + \frac{F_4 - 1}{G_1G_2G_3} + \cdots + \frac{F_n - 1}{G_1G_2\cdots G_{n-1}} \]

\[ NF_{\text{total}} = 10\log_{10} F_{\text{total}}, \] (7.17)
as is shown in section A.5. As one can see, when gain \( G_i \) is large, NF of the total system is determined by the NF of the first amplifier. In our detecting scheme, it corresponds to the amplifier of the first down converter. Therefore, we checked this first amplifier, and also the total system to evaluate the spec of our detecting electronics.

7.2.2 Spec of converter module

The first amplifier used for IP-BPM converter is NBL00419, which spec is summarized in table 7.1. These properties were checked by simple measurements.

**Gain** 6.426 GHz RF signal, generated by a signal generator, was input to the amplifier. While changing output power of the generator, output from the amplifier was monitored by a spectrum analyzer. The input-output curve is shown in Figure 7.11. The line shown in the figure is the ideal gain, expected from the spec sheet. Also, one can determine 1 dB Compression Point from Figure 7.11. 1dB Compression Point is the output power of the point where the amplifier saturates and its gain decreases by 1 dB from the expected line. From Figure 7.11, it is estimated to be 16 dB. It is shown that gain and 1 dB Compression Point are consistent with the spec sheet.

**NF** At measuring NF of the amplifier, noise of spectrum analyzer itself would be dominant if output of the amplifier was simply detected. This is because level of our concern, which is nanometer position signal, is very low. To avoid this situation, 2 low noise amplifiers were installed between the target amplifier and the spectrum analyzer to enlarge input signal. In this case, NF of 3 amplifiers would be measured, although according to Friss's formula (7.17) it is approximately equivalent to the NF of the first amplifier which we desire to measure.

When the signal generator is off, noise measured at the spectrum analyzer is considered to have originated from the first amplifier. As shown in Figure 7.12 by extrapolating the gain curve determined from signal data, the crossing point of the gain curve and the noise level is determined. This is where noise level and the input level equals, which we call equivalent noise level to input level. This is compared with the thermal noise to determine NF of the amplifier.

According to Nyquist’s Theorem, power and amplitude of thermal noise are represented as

\[ P_{TN} = k_B T \Delta f \quad (W) \]
\[ V_{TN} = \sqrt{4k_B T Z \Delta f} \quad (V), \] (7.18)

where \( k_B \) is the Boltzmann Constant, \( T \) (K) is the absolute temperature, \( Z \) (Ω) is the detecting impedance, and \( \Delta f \) (Hz) is the detecting band width. Coefficient 4 is dropped for \( P_{TN} \) to consider effective value. This is \(-174\) dBm/Hz, considering normal temperature.

Measured equivalent noise level to input level is summarized in Table 7.2. The RBW (Resolution Band Width) of the spectrum analyzer was set to be 100 Hz at the measurement, and they are calibrated to 1 Hz. Also, 3 dB is added to correct the RBW to effective band width. It is shown that the measured NF is consistent with the spec sheet.
7 DETECTING SCHEME

Figure 7.11: Gain curve of the first amplifier

Figure 7.12: Equivalent noise measurement of the first amplifier, from [15]

| Measured equivalent noise level (dBm/100Hz) | -155.4 |
| Equivalent Noise Level (dBm/Hz)          | -172.4 |
| Thermal Noise Level (dBm/Hz)             | -174   |
| NF (dB)                                  | 1.6    |

Table 7.2: Noise Figure of NBL00419

Figure 7.13: Schematics of the noise figure measurement

7.2.3 Spec of the total system

Noise figure of the total system was also measured, under a setup similar to actual beam tests. The converter module was installed in the DR, and after transmitting signal by long cables to the “eel’s bedroom”, it was detected by the phase detector. Output of the phase detector was monitored by an oscilloscope. The scheme is shown in Figure 7.13.

By giving a very small frequency difference between the generated signal and L.O. signal, output of phase detector would form a sin wave. As the generated signal becomes weaker, the sin wave would sink down into noise. One can evaluate the detection limit using this principle.

The output amplitude was measured at the oscilloscope, which is determined by the rms of output voltage. Detecting band width was 20 MHz, which correspond to 50 nsec gate width at charged ADCs. This gate width is applied at actual beam tests for data taking.

The results are shown in Figure 7.14 and Figure 7.15 which correspond to X (5.712 GHz) and
Y (6.426 GHz), respectively. Noise level is measured under condition of no input signal from the signal generator. By extrapolating the signal data, crossing point of the gain curve and the noise level is determined. This is the equivalent noise level to input level, and is compared with the thermal noise to determine NF.

The equivalent noise level is approximately $-95$ dBm for both X and Y. Since they were measured at 20 MHz band width, calibrated to 1 Hz they are $-95 - 10\log_{10}(2 \times 10^7) = -168$ dBm/Hz. Compared with thermal noise level $-174$ dBm/Hz, NF of the total system is 6 dBm. Considering noise level at phase detector effectively increasing by 3 dB due to reflection at the mixer, NF of the total system for one lane is 3 dB. This result is consistent with the NF of the first amplifier.

Applying simulation values of resonant frequencies and R/Q, 1 nm signal output amplitude calculated using (4.19) is $-102.7$ dBm and $-94.9$ dBm in horizontal and vertical, respectively. Therefore our electronics were proved to have the spec to detect 1 nm signal in vertical.
8 IP-BPM Installation

After basic parameters were checked, IP-BPM blocks were installed at ATF extraction line for beam tests. Schematics of ATF extraction line, near by IP-BPM is shown in Figure 8.1. In this section, installation of the IP-BPM body and detecting electronics is discussed in detail.

![Layout of ATF extraction line near by IP-BPM](image)

Figure 8.1: Layout of ATF extraction line near by IP-BPM, from [5]

8.1 IP-BPM Cavity

Initially one IP-BPM block was planned to be installed on a piezo actuator, while the other to be fixed on a stage. However, feedback of our piezo actuator seemed to be not working, and was noisy even in stable condition. Therefore both blocks were aligned on a common stage without precise movers, and calibration was done by sweeping beam orbit against the cavities.

In order to achieve nano-meter resolution, environmental effects must be removed, such as temperature variation and current of air in the DR. In this purpose, the IP-BPM system is installed in a plastic hut, which is covered with a vinyl sheet and triple layer of aluminum sheets. These coverages are expected to reduce the effect of temperature variation and air flow.

Thermal expansion coefficient of a material \( \alpha \) (1/K) is defined as

\[
\alpha = \frac{1}{L_0} \frac{dL}{dT},
\]

where \( L_0 \) is the original length at 0 degree, and \( L \) is the length at temperature \( T \). The expansion coefficient of the IP-BPM copper is \( \alpha_{Cu} \approx 1.7 \times 10^{-5} \). Assuming a linear thermal expansion,

\[
\Delta L = L_0 \alpha_{Cu} \Delta T
\]

is obtained. Applying typical scale of IP-BPM block length, 0.1 m, temperature shift of IP-BPM \( \Delta T < 10^{-3} \) is required in order to suppress cavity length change under 1 nano-meter.

Temperature variation was monitored to evaluate its effect to the installed IP-BPM, and the results are shown in Figure 8.2. Considering operation time required at ATF2 for calibration, approximately 1 hour, temperature drift in the DR was within 0.5°C. Temperature drift inside the hut was suppressed within 0.03°C, which is one order better compared to DR. Only 0.01°C drift was observed at surface of IP-BPM. However resolution of the heat gauge was 0.01°C, and there is still possibility of temperature drift in the order of its resolution.
Actually when somebody entered the DR while monitoring, temperature drift observed in the Ring was 2° ±C, while it was 0.7° ±C and 0.1° ±C at plastic hut and IP-BPM surface, respectively. Assuming this ratio, ΔT < 10⁻³ is not able to be achieved at IP-BPM surface, and therefore temperature drift must be monitored during operation.

### 8.2 Electronics

Since C band RF signal is easily lost while transmitting, the first down converter system was installed just near by IP-BPM plastic hut. After the down conversion, 714 MHz sensor and reference signals were transmitted to the "eel’s bedroom". At the eel’s bedroom, 714 MHz sensor signal is detected using the reference signal at phase detector, and their outputs are input to 14 bit charge ADCs. Details of these electronics were discussed in section 7.1 already.

Gain of electronics is also affected form temperature drift. Therefore, output power of the total system and temperatures of down converter, phase detector were also monitored. At this point, dummy signal of 6.426 GHz CW was input to the first down converter. The results are shown in Figure 8.2 and Figure 8.3 and is shown that temperature and gain of electronics are stable during the measurement time. However, when 2°C drift was given to the Ring, 0.7°C drift was observed in the converter and its gain had a discontinuous point. Of course during actual beam operation, no one is entering the Ring and thus no such temperature drift would occur. Nevertheless temperature of the electronics was also monitored during measurements.
9 Beam Tests

Beam tests were done at ATF extraction line, to measure position sensitivity and angle sensitivity of IP-BPM. As described in section 5.2, they are the essential characteristics of IP-BPM. Beam positions at the cavities were monitored by stripline BPMs installed near by, and their responses were calibrated by using wire scanners in advance. Beam charge at the extraction line was monitored by reference cavity, which was calibrated in relation to ICT monitor. ICT monitor response was also checked in our measurement.

In this section, modules such as beam monitors used for IP-BPM studies are described at first. Then, calibration method and results of stripline BPMs and ICT monitors are discussed in detail. Finally, measurement of important properties of IP-BPM, such as position sensitivity and angle sensitivity, are discussed in detail.

9.1 Monitors used at IP-BPM study

9.1.1 Stripline BPMs

Stripline BPMs are installed at downstream of every quadrupole magnets in the extraction line. There are two pairs of electrodes, one in horizontal and the other in vertical, aligned across to each other, as shown in Figure 9.1. When beam passes the BPM, electric charge is induced at each electrode. Since this induced current is a bipolar signal, it is first processed by clipping modules, and then read out by 14 bit charge ADC. A clipping module is a circuit consists of schottky diode, which cut off the positive part of the input signal and output the negative part only.

Beam position can be determined by the following in first order approximation:

\[
x = S_x \frac{V_2 - V_4}{V_2 + V_4} \\
y = S_y \frac{V_1 - V_3}{V_1 + V_3}
\]

where \(V_1, V_3\) are the induced voltage at vertical electrodes, and \(V_2, V_4\) are those of horizontal ones. \(S\) is a scale factor.

![Figure 9.1: Schematics of stripline BPM](image)

9.1.2 Wire Scanners

Wire scanner is a beam profile monitor. Electron beam is scanned by a tungsten wire in a particular direction, and bremsstrahlung gamma ray is emitted from the collision between beam...
and tungsten. By monitoring intensity of the gamma ray by an Air Cherenkov Counter, beam profile is acquired. The tungsten wire is aligned in 45 degrees inclined direction to horizontal or vertical, and is controlled by a stepping motor, which step is 0.5 µm. The resolution of wire scanner is limited by wire width. In our measurement, diameter of the wire was 10 µm.

### 9.1.3 ICT Monitor

ICT (Integrated Current Transformer) is used for measuring absolute beam charge in a bunch train. ICT converts the beam current to induced voltage at a pick-up coil. In case of short pulse width signal, typically ~10 ns, output signal would be unipolar with no reflection. Therefore in case of such signals, the integrated output voltage would not change even when pulse width was changed. Pulse width of ATF beam is typically 28 ps. In case of long pulse signals, reflection is seen in output signals and cannot determine the absolute value of output signal.

Please notice that the unit of ICT is $10^{10} (e^-/\text{bunch})$, which is 1.6 nC.

### 9.2 Stripline BPM Calibration

The alignment of the steering magnets, wire scanners, and stripline BPMs at ATF Extraction Line are summarized in Table 9.1. "s" shows the position of each component in the beam direction. There are systematic errors of ±5 mm for the position of each IP-BPM cavities, since their positions were measured by using rulers. The stripline BPMs used for IP-BPM study are ML11X, ML12X and ML13X, which are located just before and after the IP-BPM cavities. Before 2007 spring, ML12X and ML13X were used. After 2007 spring, ML11X and ML12X were used, since ML13X was removed due to construction of ATF2 beam line. Since beam position at IP-BPM was monitored in relation to these stripline BPMs, their precisions directly affect the precision of IP-BPM study. Thus their responses were checked by wire scanners. In this section, method and results of stripline BPM calibration are discussed in detail. Please notice that only vertical direction is discussed in the following, but horizontal direction was also calibrated by the same method.

Schematics of the stripline calibration setup are shown in Figure 9.4. Since electron beam orbit is bended proportionally to the current of steering magnets, beam displacement would be proportional to magnet current transition, at downstream of the magnet. By measuring beam displacement at a wire scanner located downstream of the magnet while changing magnet current, one can evaluate the beam displacement normalized by magnet current at the position of the considering wire scanner. Moreover, sufficiently downstream of the magnet, beam displacement would increase proportional to the distance from the magnet. Therefore by using multiple wire scanners, linear slope between beam displacement normalized by magnet current and distance from the magnet can be obtained. By extrapolating this slope to the position of each stripline BPMs, we can obtain the expected beam displacement at each stripline BPMs. By comparing
Table 9.1: Components near by IP-BPM at ATF extraction line

<table>
<thead>
<tr>
<th>Component</th>
<th>Component #</th>
<th>s (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>steering magnet (vertical)</td>
<td>ZV7X</td>
<td>25.229</td>
</tr>
<tr>
<td>steering magnet (horizontal)</td>
<td>ZH4X</td>
<td>25.923</td>
</tr>
<tr>
<td>stripline BPM</td>
<td>ML9X</td>
<td>26.464</td>
</tr>
<tr>
<td>steering magnet (vertical)</td>
<td>ZV8X</td>
<td>26.804</td>
</tr>
<tr>
<td>steering magnet (horizontal)</td>
<td>ZH5X</td>
<td>28.181</td>
</tr>
<tr>
<td>stripline BPM</td>
<td>ML10X</td>
<td>28.724</td>
</tr>
<tr>
<td>wire scanner</td>
<td>MW10X</td>
<td>29.304</td>
</tr>
<tr>
<td>steering magnet (vertical)</td>
<td>ZV9X</td>
<td>30.434</td>
</tr>
<tr>
<td>stripline BPM</td>
<td>ML11X</td>
<td>30.924</td>
</tr>
<tr>
<td>wire scanner</td>
<td>MW2X</td>
<td>31.169</td>
</tr>
<tr>
<td>stripline BPM</td>
<td>ML12X</td>
<td>33.124</td>
</tr>
<tr>
<td>wire scanner</td>
<td>MW3X</td>
<td>33.384</td>
</tr>
<tr>
<td>IP-BPM</td>
<td>BPM1</td>
<td>34.107</td>
</tr>
<tr>
<td>IP-BPM</td>
<td>BPM2</td>
<td>34.271</td>
</tr>
<tr>
<td>IP-BPM</td>
<td>BPM3</td>
<td>34.347</td>
</tr>
<tr>
<td>strip line BPM</td>
<td>ML13X</td>
<td>35.324</td>
</tr>
<tr>
<td>wire scanner</td>
<td>MW4X</td>
<td>35.564</td>
</tr>
<tr>
<td>strip line BPM</td>
<td>ML14X</td>
<td>37.909</td>
</tr>
</tbody>
</table>

Figure 9.4: Stripline calibration setup

In the actual stripline BPM response to magnet current with the expected values, we can obtain the correction factor of each stripline BPMs.

We used 4 wire scanners, MW1X, MW2X, MW3X, and MW4X installed in ATF extraction line. Figure 9.5, 9.6 and 9.7 show the response of each wire scanner in relation to magnet current of steering magnets ZV7X, ZV8X, and ZV9X. Please notice that since each wire scanner scans the beam with a tilt of 45 degrees to vertical direction, the value shown in the figures are not the actual wire scanner response, but is divided by $\sqrt{2}$. As shown in these figures, beam displacement is proportional to each magnet current. The rms of the wire scanner signal is considered to be the beam size.

In Figure 9.8, these slopes are plotted against the distance from each magnet. As shown in the figure, beam position is proportional to the distance from the magnets. These slopes are summarized in Table 9.2. Vertical errors originate from the fitting errors of the slope in Figure 9.5, 9.6 and 9.7. According to the design value of ATF, beam displacement against the magnet current $\alpha$ is constant and approximately 327.1 $\mu$rad/A, which is consistent with the slopes summarized in Table 9.2, considering those slopes correspond to $\tan \alpha \simeq \alpha$. 
**Table 9.2:** Measured slopes between beam displacement and steering magnet current

<table>
<thead>
<tr>
<th>Steering magnet</th>
<th>slope ($\mu$m/A)</th>
<th>standard error ($\mu$m/A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZV7X</td>
<td>333.725</td>
<td>5.197</td>
</tr>
<tr>
<td>ZV8X</td>
<td>336.672</td>
<td>1.836</td>
</tr>
<tr>
<td>ZV9X</td>
<td>342.393</td>
<td>7.253</td>
</tr>
</tbody>
</table>
Figure 9.9: Stripline BPM response against ZV7X current

Figure 9.10: Stripline BPM response against ZV8X current

Figure 9.11: Stripline BPM response against ZV9X current

Figure 9.12: Correction factor of Stripline BPMs

From wire scanner data shown in Figure 9.5, 9.6 and 9.7 we could obtain the expected slope at each stripline BPM position. We defined the ratio of measured beam displacement to expected beam displacement at each stripline BPMs as their correction factors. Beam displacements at ML11X, ML12X and ML13X against magnet current are shown in Figure 9.9 to Figure 9.11. Vertical errors are statistical errors. The measurement and expected values of beam displacement at each stripline BPM are summarized in Table 9.3. Also, the measurement value to expected value ratios are shown in Figure 9.12. The correction factor of each stripline BPMs is the average of the ratio weighted by statistical errors. They are summarized in Table 9.4.

In case of IP-BPM study, we used these correction factors to calibrate the beam position signal of stripline BPMs. ML11X, ML12X and ML13X are designed to be same, but somehow their correction factors were different. Mechanic alignment could be the origin of this difference.

### 9.3 ICT calibration

ICT of ATF extraction line was also calibrated. Monitoring ICT at ATF control room, sometimes ICT of extraction line increases compared to DCCT of DR. Of course beam charge cannot increase down stream, therefore calibration of ICT was considered to be necessary. Schematics of the calibration measurement are shown in Figure 9.13. A dummy pulse locked to
<table>
<thead>
<tr>
<th>Steering Magnet</th>
<th>Stripline BPM</th>
<th>measured slope (µm/A)</th>
<th>error (µm/A)</th>
<th>expected slope (µm/A)</th>
<th>error (µm/A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZV7X ML11X</td>
<td>2789</td>
<td>59.8</td>
<td>1858</td>
<td>42.6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ML12X</td>
<td>3815</td>
<td>2592</td>
<td>51.2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ML13X</td>
<td>3734.2</td>
<td>3262</td>
<td>60.8</td>
<td></td>
</tr>
<tr>
<td>ZV8X ML11X</td>
<td>2597</td>
<td>570</td>
<td>1361</td>
<td>11.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ML12X</td>
<td>3110</td>
<td>2102</td>
<td>14.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ML13X</td>
<td>2952</td>
<td>2842</td>
<td>17.9</td>
<td></td>
</tr>
<tr>
<td>ZV9X ML12X</td>
<td>1403</td>
<td>15.1</td>
<td>896</td>
<td>28.9</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ML13X</td>
<td>1909</td>
<td>1649</td>
<td>41.4</td>
<td></td>
</tr>
</tbody>
</table>

Table 9.3: Measured and expected slope of stripline BPMs against steering magnets

<table>
<thead>
<tr>
<th>Stripline BPM</th>
<th>Correction factor</th>
<th>standard error</th>
</tr>
</thead>
<tbody>
<tr>
<td>ML11X</td>
<td>1.5065</td>
<td>0.04525</td>
</tr>
<tr>
<td>ML12X</td>
<td>1.4899</td>
<td>0.02358</td>
</tr>
<tr>
<td>ML13X</td>
<td>1.0881</td>
<td>0.03806</td>
</tr>
</tbody>
</table>

Table 9.4: Correction factors of stripline BPMs in comparison with wire scanners

DR orbiting frequency, 2 MHz, is generated by a function generator. Pulse width is 30 ns, and intensity can be controlled by changing the pulse height. While this dummy pulse passes through the ICT monitor, transmitted beam charge was monitored by an oscilloscope. By relating the actual transmitted beam charge to ICT read out, ICT can be calibrated to beam charge.

To determine the charge of dummy pulse, waveform of the transmission was recorded by the oscilloscope. By integrating the voltage, total transmitted voltage is obtained. By dividing impedance Z = 50 Ω, total beam charge is obtained. To calculate ICT, total beam charge is normalized by 1.6 nC.

Results are shown in Figure 9.14. Abscissa is the actual transmitted beam charge monitored at oscilloscope, and ordinate is the read out of ICT monitor, which is observed at ATF control room. As shown in the figure, ICT read out is over estimating the beam charge for approximately 10 %.

Actually we wanted to try with much shorter pulse width, since ATF bunch length is approximately 28 ps. Also, ICT read out might have reflection in case of 30 ns pulse width. Unfortunately our function generator could not generate such short pulse. There was also another pulse generator which could generate shorter pulse of 500 ps, but could not apply external trigger, which is DR orbiting 2 MHz. To confirm our result, we also recorded the ICT read out by an oscilloscope for each pulse generator case. The waveform of ICT read out was also integrated, which must be proportional to ICT integrated at the ADC. By relating the integrated ICT to transmitted pulse charge from each pulse generator, we confirmed that the result is independent of pulse width, as shown in Figure 9.15. Therefore our result is considered to be reliable.

In our beam tests, ICT is calibrated by using the fitting function shown in Figure 9.14.
9.4 Position Sensitivity

As discussed in section 5.2, ultra high position sensitivity is required for IP-BPM.

9.4.1 Theoretical Prediction

As shown in (4.19), output amplitude of IP-BPM position signal can be calculated as

$$V_{out0} = \sqrt{ZF} = \frac{\omega q}{2} \sqrt{\frac{Z}{Q_{ext}}} (R/Q) \exp \left( -\frac{\omega^2 \sigma_z^2}{2 \sigma^2} \right),$$

(9.3)

where $\omega$ is the resonant angular frequency, $q$ is the beam charge, $Z$ is the detecting impedance, and $\sigma_z$ is the bunch length in z direction. Applying simulated parameters from Table 6.1 and results of $R/Q$ simulation, nominal ATF2 charge $q = 1.6$ nC and typical bunch length $\sigma_z = 8$ mm, the expected outputs are approximately 1.63 mV/µm and 4.02 mV/µm, in X and Y, respectively. They correspond to approximately $-102.7$ dBm and $-94.9$ dBm output power for 1 nm offset beam. Since the detection limit of the electronics was $-95$ dBm as discussed in section 7.2, IP-BPM is able to detect 1nm signal in vertical, and 3 nm signal in horizontal.

9.4.2 Measurement Scheme

Schematics of the measurement are shown in Figure 9.16. By controlling the current of steering magnets, electron beam was swept in horizontal or vertical against the cavity. Output signals from
opposite ports were combined at the combiner, and its signal amplitude was detected by diode. Signal amplitude was monitored by an oscilloscope at the "eel's bedroom". Beam position at the cavity was monitored by stripline BPMs, and the beam position at each cavity was interpolated from them.

Also to check the symmetry of the cavity and effect of combiner, position sensitivity was checked for 1 port case also. In this case, only 1 port of the direction was used to detect the amplitude, while the other port of the cavity and combiner were terminated by 50 Ω. Since output signal from the opposite port is considered to be opposite in phase and same in amplitude, compared with 1 port case the sensitivity of 2 port case is expected to be doubled.

9.4.3 Results

Since signal amplitude is measured ignoring the phase information, signal response to beam position forms a V shape, with its minimum at the electrical center of the cavity. It is shown in Figure 9.17. Ordinate is the signal amplitude, normalized at nominal beam charge of ATF2, 1.6 nC/bunch. Abscissa is the beam position interpolated from stripline BPM information. Slopes of the V shape correspond to position sensitivity, and they are summarized in Table 9.5. Since they are normalized to the same condition with prediction, they can be easily compared. Lines in the figures show the expected V slope.

Measured power is approximately 4 dB worse compared to expected power both in horizontal and vertical. However, it is still possible to measure 2 nm position signal in vertical and 4 nm position signal in horizontal. This is enough spec to achieve the goal of ATF2.

9.4.4 Discussions

The 4 dB difference between measured and expected power is considered to be originated from inaccurate diode calibration. In position sensitivity measurement, a diode was used to detect the signal, in order to obtain the absolute value of the output signal power. In order to calibrate the detected output amplitude to the signal amplitude at cavity port, calibration was made by inputting constant wave of known power to the measurement scheme. However, there are two factors which can affect the accuracy of calibration.

First one is the signal rise time of the diode. Monitored beam signal had a finite rise time of approximately 15 ns. Since sensor signal is a pulse, the observed pulse height is reduced compared with constant wave signal of same amplitude, as shown in Figure 9.18. However, we could not generate such a quick rise pulse by our signal generator and therefore cannot measure its actual loss. Therefore, a circuit simulator called Circuit Maker was used to evaluate the loss of
9 BEAM TESTS

Figure 9.17: Output signal voltage vs beam position

![Diagram showing output signal voltage vs beam position]

Figure 9.18: Effect of finite rise time at the diode detector

![Diagram showing rise time and pulse height]

pulse height. Applying the equivalent circuit of our diode\cite{16} and optimizing the parameters to reproduce the measured signal rise time, 2 dB loss between CW and pulse signal was observed.

Second one is the inaccuracy of the spectrum analyzer used to measure the input power from the signal generator. There are three inaccuracies in case of absolute power measurement. Firstly, there is uncertainty in absolute self calibration. According to the spec sheet of our spectrum analyzer\cite{17}, this is ±0.15 dB. Next, there is uncertainty of the gain of amplifier, which is used to measure power level different from the reference level. This is ±0.6 dB at maximum in our case. Finally, there is uncertainty of the frequency response, which originate from imperfect flatness among the measurement band width. This is ±2.5 dB in our case. Therefore, inaccuracy of the spectrum analyzer is estimated to be ±3.25 dB.

Considering both inaccuracies, 4 dB difference seems to be reasonable.

9.5 Angle Sensitivity

Reducing angle sensitivity is critical for IP-BPM, since angle jitter is expected to be large at IP, due to strong final focus. To be more precise, approximately 400 µrad angle jitter in maximum is expected at the IP area. Angle signal could easily saturate the detecting electronics and therefore...
narrow the dynamic range. As discussed in section 4.6.1, reducing angle sensitivity can be achieved by designing a small cavity length $L$. $L$ of IP-BPM is 6 mm, while other ATF2 Q-BPMs are 12 mm.

The origin of angle signal is the phase transition while the beam bunch passes the cavity. Since HFSS doesn’t consider the effect of RF phase, HFSS simulation cannot be applied. Angle sensitivity of IP-BPM cavity was measured by the following measurement.

### 9.5.1 Prediction from theory

Angle sensitivity of a cavity is characterized with what called the "equivalent position signal". This is defined by comparing the angle signal with position signal of same amplitude. Assuming (4.64), it is simply calculated as

$$\frac{|V_{\text{angle}}|}{|V_{\text{position}}|} \sim \frac{\omega L^2 x'}{8\sqrt{2}c x} = 1 \iff x = \frac{\omega L^2}{8\sqrt{2}c} x'$$

(9.4)

Although it is difficult to determine effective cavity length $L_{\text{eff}}$, but assuming to be the longitudinal length where field strength is over $1/e$ of the peak, it is 9 mm and 8 mm for X and Y, respectively. These were determined from Figure 6.11. Applying these values, equivalent position signals are expected to be 0.86 $\mu$m/mrad and 0.76 $\mu$m/mrad for X and Y, respectively.

### 9.5.2 Measurement Scheme

Instead of tilting electron beam orbit against IP-BPM cavity, the cavity was forced to be tilted mechanically against beam orbit, by clipping spacers between BPM block and the stage. This is because in comparison with angle jitter of the beam, inclination angle must have been changed drastically, which was beyond beam orbit tuning. By changing steering magnet current, beam was swept in horizontal or vertical direction, and the output amplitude was detected by diode. Beam position at the cavity was interpolated from stripline BPMs. The whole scheme is shown in Figure 9.19.

In case of position sensitivity measurement, the signal response to beam displacement formed a V shape, with its minimum being zero. However for angle signal case, the amplitude is none zero even at cavity center, due to the angle signal. As discussed in section 4.6.1, angle signal and position signal are 90 degrees different in phase, which are separated in $I$ signal and $Q$ signal. Since output amplitude corresponds to $V = \sqrt{I^2 + Q^2}$, $V$ is not zero even at $I = 0$. The angle slope forms a smooth curve at center, due to the expression as previously noted. In our measurement, this minimum signal $V = Q$ was compared with the position signal equivalent in amplitude, which corresponds to the equivalent position signal, as described in Figure 9.20.
9.5.3 Results

The signal response in various situations of cavity angle against the beam is shown in Figure 9.21 and Figure 9.22. Considering pedestal (approximately 500 ADC count), signals are still non-zero at cavity center as predicted. Actually, initial alignment in horizontal turned out to have angle against the beam line. Therefore, the minimum signal shown in Figure 9.21 is not angle 0, but 12.8 mrad. Also, the measured slope has a smooth curve in center, even in non-tilted case. This is due to the characteristics of the diode used for signal detection.

In order to determine the equivalent position signal, the non-tilted case was fitted applying second order polynomial function for the smooth curve region, and linear function for the linear region. Then, base line shift of tilted cases were converted to equivalent position signal from these fitting functions.

Their equivalent position signals are plotted against beam angle in Figure 9.23. The equivalent position signals at the center were estimated to be approximately 1.46 µm/mrad, 3.24 µm/mrad for horizontal and vertical, respectively. Compared with expected values, angle sensitivity is not reduced enough. However, expected values change due to definition of $L_{eff}$. To reproduce measurement results, $L_{eff}$ is 11.8 mm and 16.5 mm for X and Y respectively, which are still not impossible referring Figure 6.11.

Considering maximum beam jitter at the IP, 0.4 mrad, equivalent position signal would be 0.59 µm and 1.29 µm in horizontal and vertical, respectively. As discussed later in section 10, dynamic range of the detecting electronics is approximately 5 µm, which would not saturate in case of strong angle jitter, due to this angle sensitivity reduction.
Figure 9.23: Angle sensitivity of IP-BPM
10 Position Resolution Measurement (Single Bunch)

Finally, position resolution of IP-BPM was measured by the following method. It goes without saying that, in order to meet ATF2 requirements, IP-BPM must have nano-meter position resolution, which has not been achieved in BPM history. Position resolution measurement was carried out in vertical direction, since the goal is much more strict compared to horizontal.

10.1 Principle

We used 3 cavities for position resolution measurement. As discussed in section 5.1, 3 cavities are needed to determine position resolution. This is to measure the residual between beam orbit determined by 2 cavities and 1 more beam position from the 3rd. From upstream, they are called BPM1, BPM2, and BPM3. Actually we have 4 cavities fabricated, but we chose 3 cavities whose resonant frequencies correspond well. BPM1, BPM2 and BPM3 correspond to 1-1, 2-1, 2-2 shown in Table 6.2, respectively.

Position resolution is defined as "RMS of the residual between measured and predicted beam position at the center cavity (BPM2)" × "Geometrical Factor". Predicted beam position is calculated from the information of BPM1 and BPM3. Geometrical Factor is a factor to correct the propagation of the error due to the alignment of the 3 cavities to calculate resolution of a single cavity.

Position resolution measurement was carried out at ATF Extraction Line, same as other measurements. The measurement is consisting of the following steps:

1. Reference Cavity Calibration, to calibrate the reference cavity signals to beam charge in relation to ICT monitor.
2. I-Q Tuning, to separate position signal to I and other noise components to Q.
3. Calibration Run, to calibrate sensor cavity signals to actual beam position, using stripline BPMs.
4. Resolution Run, to measure the RMS of the residual between measured and predicted beam position at BPM2.

As written above, Resolution Run is operated to determine the RMS of the residual between measured and predicted beam position at BPM2, using the information from BPM1 and BPM3. It is carried out for about 1000 events typically. Calibration Run is operated to calibrate the RMS measured at Resolution Run to the actual beam displacement. By controlling the current of steering magnets, beam is swept against the cavities. For each beam orbit, typically 50 events were taken. Beam positions at each cavities were interpolated from the response of stripline BPMs. At this point, position sensitivities of the sensor signals were reduced by a variable attenuator. This is to reduce position sensitivities of the cavities to stripline BPM level, and also to widen the dynamic range of the detection electronics in order not to saturate during the calibration run.

In this section, method of position resolution measurement is described in details.

10.2 Reference Cavity Calibration

The nominal charge of ATF electron beam is $1.0 \times 1.6 \text{ nC (}10^{10}\text{e}^{-}/\text{bunch}). However, it is difficult to make a stable nominal charge beam operation at present ATF beam line. Instead, we made our measurement in the range of $0.6 (< ICT < 0.7 (< 1.6 \text{ nC}). To monitor beam charge and apply this data cut, we used the reference cavity. Monopole mode of the reference cavity is designed to have the same resonant frequencies to those of dipole modes of the sensor cavity. The reference signal is not sensitive to beam position, but proportional to beam charge. In advance of position resolution measurement, we calibrated the reference cavity signal to actual beam charge using ICT.

ICT was changed by controlling the laser power of the RF gun at ATF injection, and amplitude of the reference cavity was monitored at each ICT value. The result is shown in Figure 10.1, which
shows a linear correlation between reference signal and ICT. We made a linear fit, and used this function to calibrate the reference signal to actual beam charge. Also, applied data cut is shown in blue region in the figure.

10.3 I-Q tuning

The method of I-Q tuning is the same as discussed in section 7.1. However, I-Q tuning in case of position resolution measurement was carried out very carefully, as discussed in the following.

Experience shows that, we could not expect good position resolution when Q signal, not only I signal, saturates. This happens for example when beam angle is large. Due to megascopic phase tuning using an oscilloscope, contamination of noise components to the position signal is inevitable. Therefore large angle signals could easily make the electronics saturate, in case of both I and Q. To avoid these effects, IP-BPM is designed to have low angle sensitivity as discussed in section 5.3.

To improve precision of phase tuning, feedback using the beam is effective. To minimize beam angle against the cavity, for example conditions

1. Heights of the 3 cavities are equated
2. Electron beam passes the cavity center of the 3 cavities (I signals become zero) at the same time

are required.

At first, we reduced the position sensitivity using the variable attenuator, and made a phase tuning for each cavities by eyes, using oscilloscopes. Since position sensitivity and, of course angle sensitivity are reduced, the signal-noise separation is precise at this range of low sensitivity. Then, as shown in Figure 10.2 (b), we searched for a beam orbit which makes all Q signals zero, by sweeping the beam against cavities. In this situation, beam orbit is expected to be parallel to 3 sensor cavities. Since I signals correspond to the beam position at each cavities, we can determine their relative position displacement from amplitude difference of their I signals. We controlled cavity height by clipping spacers between BPM blocks and stages to minimize the I signal difference, as shown in Figure 10.2 (c). We repeated this procedure at lower attenuation.

Finally, we succeeded to align 3 cavities at the same height in precision of a few microns. In this condition, precise I-Q tuning is performable since Q components are commonly small at every 3 cavities and is easier to maximize the I signal sensitivity to beam displacement.
10.4 Calibration Run

The purpose of "Calibration Run" is to calibrate the sensor cavity response to actual beam position. As shown in Figure 10.3, electron beam is swept against sensor cavities by controlling magnet current and response of the sensor cavities are monitored. Beam positions at each cavities were interpolated from stripline BPMs.

In this situation, it is very important to sweep the beam parallel to 3 cavities without shifting the Q signals, or, changing beam orbit angle. We achieved this by sweeping the beam using two steering magnets, as shown in Figure 10.3.

Calibration Run was made under 40 dB, 30 dB, 20 dB attenuation cases. Attenuation was
controlled by the variable attenuator. This is to enlarge dynamic range of the electronics, in order not to saturate while sweeping the beam. Also, effect of beam position jitter is expected to become negligible in these cases, due to low position sensitivity. In 10 dB and 0 dB attenuation case, Calibration Run could not be operated since beam would be out of dynamic range even when it is swept in minimum magnet current step. Therefore, calibration slope for these cases were estimated by extrapolating the results of 30 dB attenuation case. Calibration Run was operated three times for each attenuation case, to estimate their systematic errors.

Figure 10.4, 10.5, and 10.6 show the results of Calibration Run of the 3 cavities under 30 dB attenuation. Each data point is consist of typically 50 events. Abscissa is the beam position at each cavity, which are interpolated from stripline ML12X and ML13X. Ordinate is the ADC count of I (position) signal of each sensor cavity. Vertical errors are statistical errors of I signal and errors from stripline BPMs. Horizontal errors are systematic error of cavity position, due to megascopic measurement using rulers. Horizontal errors of different beam positions have strong correlations, and they are not random. Therefore we obtained calibration slope at both edges of the systematic error, and considered its width to be the systematic error of the slope.

Calibration slope for calibrating the I signal to actual beam position is summarized in Table 10.1. Since signal amplitude is proportional to square root of signal power, the slopes are expected to shift for factor $\sqrt{10}$ at 10 dB shift of variable attenuator, which is consistent with the results.
Table 10.1: Measured calibration slope of each cavities

<table>
<thead>
<tr>
<th>Attenuation</th>
<th>Slope (ch/µm)</th>
<th>BPM1</th>
<th>BPM2</th>
<th>BPM3</th>
</tr>
</thead>
<tbody>
<tr>
<td>40dB</td>
<td>18.4245</td>
<td>20.1671</td>
<td>19.7056</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Stat. Error (ch/µm)</td>
<td>0.2056</td>
<td>0.2736</td>
<td>0.2595</td>
</tr>
<tr>
<td></td>
<td>Sys. Error (ch/µm)</td>
<td>0.0204</td>
<td>0.0256</td>
<td>0.0264</td>
</tr>
<tr>
<td>30dB</td>
<td>62.4153</td>
<td>63.7572</td>
<td>63.0263</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Stat. Error (ch/µm)</td>
<td>0.8057</td>
<td>0.5543</td>
<td>0.2385</td>
</tr>
<tr>
<td></td>
<td>Sys. Error (ch/µm)</td>
<td>0.1673</td>
<td>0.1680</td>
<td>0.1644</td>
</tr>
<tr>
<td>20dB</td>
<td>182.941</td>
<td>191.596</td>
<td>188.462</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Stat. Error (ch/µm)</td>
<td>4.832</td>
<td>5.190</td>
<td>4.192</td>
</tr>
<tr>
<td></td>
<td>Sys. Error (ch/µm)</td>
<td>0.559</td>
<td>0.597</td>
<td>0.588</td>
</tr>
</tbody>
</table>

As discussed, we used the calibration slope extrapolated from 30dB attenuation data to calibrate Resolution Run results under 10 dB and 0 dB attenuation case, as shown in Figure [10.7]. The solid lines show the expected sensitivity extrapolated from 30 dB data, while measured values are shown as points. Under 20 dB attenuation, we have fewer data points due to existence of out-of-range beam, and also there is larger contribution of beam jitter due to larger position sensitivity, as shown in Table [10.1].

### 10.5 Resolution Run

Calibration Run was operated for approximately 50 events typically, at each beam orbit. At each Resolution Run, typically 200 events were taken for a fixed beam orbit, to evaluate position resolution. Resolution Run was taken for 9 sets, for each of 30 dB, 20 dB, 10 dB, 0 dB attenuation cases. Under 0dB attenuation case, a long run of 1 hour was also taken.

The purpose of Resolution Run is to measure the "rms of the residual between measured and predicted beam position at the center cavity (BPM2)". The predicted beam position was obtained from a linear regression analysis, using information from BPM1 and BPM3 as parameters. Those are,

- Vertical position signals (in phase components) of BPM1, BPM3: Y1I, Y3I
- Vertical noise components (out of phase components) of BPM1, BPM3: Y1Q, Y3Q
- Horizontal position signals and noise components of BPM1, BPM3: X1I, X3I, X1Q, X3Q
- Beam charge detected at X reference cavity and Y reference cavity: XREF, YREF

Specifically, we assumed a linear regression formula as below, and determined the coefficient α of each parameter.

\[
Y^{2I_{predicted}} = \alpha_0 + \alpha_{Y1I} \cdot Y1I + \alpha_{Y1Q} \cdot Y1Q + \alpha_{Y3I} \cdot Y3I \\
+ \alpha_{Y3Q} \cdot Y3Q + \alpha_{Y_{ref}} \cdot YREF + \alpha_{X1I} \cdot X1I + \alpha_{X1Q} \cdot X1Q \\
+ \alpha_{X3I} \cdot X3I + \alpha_{X3Q} \cdot X3Q + \alpha_{X_{ref}} \cdot XREF.
\]

At this point, \(Y^{2I_{predicted}}\) is the predicted beam position at BPM2. The residual is calculated as,

\[
\text{Residual} = Y^{2I_{measured}} - Y^{2I_{predicted}}.
\]
plot of measured $Y_{2I}$ vs predicted $Y_{2I}$, and the distribution of its residual is shown in left below figure. Rms of the residual, 22.02 ADC count, corresponds to the resolution of this measurement. Upper right figure shows the distribution of YREF, which corresponds to beam charge distribution during operation. Applied beam charge cut is shown in the red region. Right below figure shows the time dependency of the residual, and it is shown that resolution is stable for 1 hour. Figure 10.9 shows the correlation between the residual and each parameter of the regression formula. No particular correlation is seen to be left in Figure 10.9.

Applied data cuts are described in the following. First, since di-pole mode signal changes in relation with beam charge, beam intensity cut of $0.640 \times 1.6 \text{ (nC)} < ICT < 0.755 \times 1.6 \text{ (nC)}$, which corresponds to 6442 (ADC count) < YREF < 7532 (ADC count) was applied. Also to avoid saturation at edge of dynamic range of the electronics, 3000 (ADC count) < $Y_{1I}, Y_{2I}, Y_{3I}$ < 13000 (ADC count) for position signals is applied. Applying extrapolated calibration slope, this corresponds to 4.96 $\mu$m dynamic range.

9 sets of shorter runs (typically 200 events) were also taken, in order to estimate systematic error of Resolution Run. Measured values of their rms of residual are shown in Figure 10.10. At first we calculated the regression coefficients by using all 1,800 events, and these coefficients were applied to each data set to calculate $Y_{2I}^{predicted}$. Fluctuations of rms among the 9 sets correspond to systematic errors of the rms. Vertical error of Figure 10.10 is the statistical error, which is determined from $\text{RMS}/\sqrt{2N}$, where N is the event number. As a result of these 9 sets, rms is estimated to be 25.75 (ch) ± 1.00 (ch, sys.), and the rate of systematic error was approximately 3.88 %. Assuming this systematic error rate, rms of the 1 hour run is obtained as 22.02 (ch) ±
Figure 10.9: Correlation between residual and regression parameters left after analysis
Figure 10.10: RMS of residual measured at each set of resolution run

0.70 (ch, stat.) ± 0.86 (ch, sys.).

We have to be careful since the rms of residual measured in this scheme is not exactly the position resolution of a single cavity. To obtain position resolution of a single cavity, we need to consider what called the “Geometrical Factor”. This is a coefficient determined by positional relation of the 3 cavities. Alignment of the 3 cavities is shown in Table 9.1. Defining BPM\textsubscript{i} as the \textit{i}th cavity, \(z_{ij}\) as the distance between BPM\textsubscript{i} and BPM\textsubscript{j}, \(I_i\) as beam position signal of BPM\textsubscript{i}, and assuming their resolution \(R_i\) are all equivalent, rms of

\[
f(I_1, I_2, I_3) = I_2 - \frac{z_{23} \times I_1 + z_{12} \times I_3}{z_{31}}
\]

is measured in our study, where the first term is the measured \(I_2\) and the second term is the predicted \(I_2\), interpolated from \(I_1\) and \(I_3\).

Defining this rms as \(R_f\), Geometrical Factor is calculated as

\[
\frac{R_2}{R_f} = \frac{1}{\sqrt{\left(\frac{\partial f}{\partial I_1} R_1\right)^2 + \left(\frac{\partial f}{\partial I_2} R_2\right)^2 + \left(\frac{\partial f}{\partial I_3} R_3\right)^2}}
\]

\[
= \frac{1}{\sqrt{\left(\frac{z_{23} R_1}{z_{31} R_2}\right)^2 + 1 + \left(\frac{z_{12} R_3}{z_{31} R_2}\right)^2}} = \frac{1}{\sqrt{\left(\frac{z_{23}}{z_{31}}\right)^2 + 1 + \left(\frac{z_{12}}{z_{31}}\right)^2}}.
\]

Please note that assumption \(R_1 = R_2 = R_3\) was used. According to Table 9.1, \(z_{12} = 164\) mm, \(z_{23} = 76\) mm, and \(z_{31} = 240\) mm are obtained. Assigning these values, Geometrical Factor of our measurement is approximately 0.799.

Applying the extrapolated calibration slope for 0dB attenuation case, position resolution of a single cavity is obtained from the following:

\[
\text{Position Resolution} = \text{Geometrical Factor} \times \left(\frac{\text{RMS of residual measured at Resolution Run}}{\text{Calibration Slope measured at Calibration Run}}\right).
\]

The results are summarized in Table 10.2. Therefore the position resolution was estimated to be

\[
8.72\ (\text{nm}) \pm 0.28\ (\text{nm, stat.}) \pm 0.35\ (\text{nm, sys.}).
\]
Table 10.2: Results of position resolution measurement under no attenuation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calibration Slope (ch/μm)</td>
<td>2016</td>
</tr>
<tr>
<td>Stat. Error (ch/μm)</td>
<td>17.53</td>
</tr>
<tr>
<td>Sys. Error (ch/μm)</td>
<td>5.31</td>
</tr>
<tr>
<td>Resolution Run RMS (μm)</td>
<td>22.02</td>
</tr>
<tr>
<td>Stat. Error (μm)</td>
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</tr>
<tr>
<td>Sys. Error (μm)</td>
<td>0.86</td>
</tr>
<tr>
<td>Geometrical factor</td>
<td>0.799</td>
</tr>
<tr>
<td>Dynamic Range (μm)</td>
<td>4.96</td>
</tr>
<tr>
<td>Position Resolution (nm)</td>
<td>8.72</td>
</tr>
<tr>
<td>Stat. Error (nm)</td>
<td>0.28</td>
</tr>
<tr>
<td>Sys. Error (nm)</td>
<td>0.35</td>
</tr>
</tbody>
</table>

Figure 10.11: Measured position resolution under various attenuations

Statistical errors are those which originate from the number of statistics of Resolution Run, and systematic errors are the other errors. Thus for example, statistical error of calibration slope is a systematic error of position resolution.

Average beam charge (ICT) during 1 hour measurement was 0.68 × 1.6 nC. This implies that under nominal condition of ATF2, which is 1.00 × 1.6 nC, position resolution is expected to be 5.94 nm.

Position resolutions under 30 dB, 20 dB, 10 dB attenuation cases were measured by the same procedure. The results are shown in Figure 10.11. Solid line is the expected resolution extrapolated from 30 dB data. Measured resolution is seen to run off the expected line as attenuation reduces. This is considered to occur due to the effect of larger beam position jitter or angle jitter measured at lower attenuation.

Typical beam condition under resolution run is shown in Figure 10.12 and Figure 10.13. Figure 10.12 shows the beam position jitter at BPM2, which is the distribution of \( Y_2 I \) calibrated to beam displacement. RMS of beam jitter is approximately 7 μm. Figure 10.13 shows the beam angle jitter, which is the distribution of \( (Y_3 I - Y_1 I) \) calibrated to beam angle. Since the distance between BPM1 and BPM3 is 240 mm in beam direction, beam angle can be calculated from the difference of vertical beam position \( (Y_3 I - Y_1 I) \). RMS of angle jitter is approximately 2.4 μrad.
10 POSITION RESOLUTION MEASUREMENT (SINGLE BUNCH)

Figure 10.12: Beam position jitter at BPM2 under 20 dB attenuation

Figure 10.13: Beam angle jitter at BPM2 under 20 dB attenuation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Coefficient</th>
</tr>
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<tbody>
<tr>
<td>Y1I</td>
<td>0.312</td>
</tr>
<tr>
<td>Y1Q</td>
<td>0.132</td>
</tr>
<tr>
<td>Y3I</td>
<td>0.694</td>
</tr>
<tr>
<td>Y3Q</td>
<td>-0.029</td>
</tr>
<tr>
<td>YREF</td>
<td>0.074</td>
</tr>
<tr>
<td>X1I</td>
<td>-0.059</td>
</tr>
<tr>
<td>X1Q</td>
<td>-0.271</td>
</tr>
<tr>
<td>X3I</td>
<td>0.102</td>
</tr>
<tr>
<td>X3Q</td>
<td>-0.059</td>
</tr>
<tr>
<td>XREF</td>
<td>-0.515</td>
</tr>
</tbody>
</table>

Table 10.3: Regression coefficients of no attenuation case

10.6 Regression Parameters

Coefficients of each regression parameter at (10.1) are summarized in Table 10.3. Constant term is omitted since it does not have any physical context.

In ideal condition, beam position at BPM2 is obtained by interpolating those at BPM1 and BPM3. Therefore parameters necessary are only the I signal of BPM1 and BPM3. In this case, it is

\[
Y2I_{predicted} = \frac{z_{23}}{z_{12} + z_{23}} Y1I + \frac{z_{12}}{z_{12} + z_{23}} Y3I
= 0.317 \times Y1I + 0.683 \times Y3I,
\]

where \(z_{12} = 164\) mm, \(z_{23} = 76\) mm are the distances in z direction between BPM1 and BPM2, BPM2 and BPM3, respectively.

Using results of calibration run (Table 10.1), coefficients of Y1I and Y3I can be calibrated to actual beam position. Calibrated coefficients are shown in Table 10.4. In Table 10.4 coefficient of Y1(3)I shows the beam displacement (nm) of \(Y2I_{predicted}\) due to 1nm beam position shift at BPM1(3). Compared with (10.7), coefficients are quite similar. It shows that I-Q tuning (Position-Noise tuning) was achieved enough.

However in practice, parameters other than Y1I and Y3I are needed to predict beam position, Y2I. Those parameters are considered to originate from the following contexts.

Q Components Even when Position-Noise tuning is incomplete, there is no contribution of Q in
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Coefficient</th>
</tr>
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<tbody>
<tr>
<td>Y1I</td>
<td>0.308</td>
</tr>
<tr>
<td>Y3I</td>
<td>0.691</td>
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</tbody>
</table>

Table 10.4: Coefficients calibrated to actual beam displacement

![Diagram](image)

Figure 10.14: Misalignment between 2 cavities

principle, since $Q$ is orthogonal to $I$. However if I-Q axis are relatively misaligned among the 3 cavities, $Q$ component of BPM1, BPM3 are necessary to determine $Y2I$. In a simplified case, for example when there is a relative misalignment between two cavities as shown in Figure 10.14 to determine $Y2I$ one must calculate

$$Y2I = Y1I \cos \theta - Y1Q \sin \theta.$$ (10.8)

**X Components** Even when $Y$ direction of the cavity is not exactly aligned to the vertical direction, there is no contribution of $X$ in principle, since $X$ is orthogonal to $Y$. However like the situation of $Q$ component, if there is a relative misalignment in rotating direction among the 3 cavities, $X$ component of BPM1, BPM3 can contribute to $Y2I$.

**XREF, YREF** Contributions from these terms are natural since di-pole signals depend not only on beam position but also on beam charge. Since reference signals are sensitive to beam charge, their contributions are inevitable.

From this discussion, main terms of the regression analysis are considered to be $Y1I$ and $Y3I$, and sum of all the parameters are ideally 1. Therefore, contribution of other parameters can be evaluated from the ratio of coefficient of the considering parameter and the ideal sum of main terms, 1. The coefficients in Table 10.3 show how much ADC counts the $Y2Ipredicted$ shifts in relation to 1 ADC count change of the considering parameter. However since the parameters are shown in arbitrary units, we could not simply compare these coefficients. Thus standard regression coefficients are applied to evaluate the contribution of each parameter. Definition of standard regression coefficients are described in Appendix A.6.

Also to evaluate the contribution of each term, 6 types of regression analysis (A) \(\sim\) (F) were applied. Standard regression coefficients of the applied parameters and the resulted RMS are summarized in Table 10.5. All the parameters are used in set (A), while only main terms and YREF are used in set (B). It is shown that RMS is improved by adding parameters to set (B).

As shown in Table 10.5, the coefficients of main terms, Y1I and Y3I, are stable. However, other coefficients are varied. This is considered to happen since these parameters are not all independent, but have correlations between each other, as shown in Figure 10.15. For example there is a strong correlation between YREF and XREF since they both monitor the beam charge.
To evaluate the contribution of each parameter accurately, only one term of the pair must be applied in regression, which is the case of set (F) in Table 10.5. According to set (F), contribution from YREF is approximately 7 %, and contribution from Q and X components are approximately 2 ~ 4% respectively, compared to main terms.
10.7 Thermal Noise

Position resolution of IP-BPM is limited in principle due to thermal noise of detecting electronics. As discussed in section 7.2, Friss's Formula shows that thermal noise is determined mainly by the first amplifier. In case of IP-BPM, it is the one installed in the first down converter. To evaluate thermal noise of the converter and total system, signal from a sensor cavity was divided before the first down converter, and both signals were measured by the same detecting scheme. From the correlation of these 2 signals, thermal noise of the electronics was determined.

The detecting scheme is shown in Figure 10.16. Upper side of the figure corresponds to the upstream of beam line. During this measurement, Y signal was divided as shown in the figure, while X signal was detected by the same scheme without being divided. One signal is considered as sensor signal, while the other as reference signal. Thermal noise would correspond to the rms of the residual between measured and predicted position (I), where measured signal is the sensor signal, and the predicted is the prediction from the reference signal.

The measurement method is the same with position resolution measurements. Calibration Run was carried out in 40 dB, 30 dB, and 20 dB attenuation condition. Calibration slope for 10 dB, 0 dB conditions were extrapolated from the calibration slope of 30 dB case. Resolution Run was carried out under 30 dB, 20 dB, 10 dB and 0 dB attenuation. At 0 dB case, a long run of 20 minutes was made to measure rms of the residual. Also to estimate the systematic error of the rms, 9 sets of typically 200 events were taken.

The predicted signal is estimated by a linear regression analysis, as following:

\[ Y_{2I, \text{predicted}} = \alpha_0 + \alpha_{Y_{1I}} \cdot Y_{1I} + \alpha_{Y_{1Q}} \cdot Y_{1Q} + \alpha_{Y_{\text{REF}}} \cdot Y_{\text{REF}}. \]  

(10.9)

\( Y_{1I}, Y_{1Q} \) are the I, Q components of the reference signal, while \( Y_{2I} \) is the I component of the sensor signal. \( Y_{\text{REF}} \) is the intensity of Y reference cavity signal (not the reference signal \( Y_{1} \)). The residual is obtained by

\[ \text{Residual} = Y_{2I, \text{measured}} - Y_{2I, \text{predicted}}. \]  

(10.10)

The actual thermal noise is compared with equivalent position signal, by calculating

\[ \text{Statistical Factor} \times \left( \frac{\text{RMS of residual measured at Resolution Run}}{\text{Calibration Slope measured at Calibration Run}} \right). \]  

(10.11)

Statistical Factor originates due to using 2 divided signals, and is \( 1/\sqrt{2} \).

At analyzing, intensity cut of \( 0.640 \times 1.6 \) (nC) < ICT < \( 0.755 \times 1.6 \) (nC) was applied, which correspond to 6442 (ADC count) < \( Y_{\text{REF}} \) < 7532 (ADC count). To avoid the saturation of the electronics, 3000 (ADC count) < \( Y_{1I}, Y_{2I} \) < 13000 (ADC count) was also applied.

Results of the Calibration Run are shown in Figure 10.17, and the rms of residual measured at 9 sets of short Resolution Run are shown in Figure 10.18. The result of long Resolution Run is shown in Figure 10.19 and Figure 10.20.

From these data, thermal noise of the electronics is estimated, which is summarized in Table 10.6. As a result, thermal noise is estimated to be equivalent to the position signal of

\[ 3.86 \text{ (nm)} \pm \ 0.15 \text{ (nm, stat.)} \pm 0.25 \text{ (nm, sys.)}. \]  

(10.12)

During the measurement, average bunch charge (ICT) was \( 0.67 \times 1.6 \) nC. Since signal was divided into 2 by a 6 dB splitter, position signal is 1/2 of the nominal in amplitude. Considering the nominal bunch charge at ATF2, \( 1.00 \times 1.6 \) nC, under ideal beam condition and by eliminating noise other than thermal ones by analysis, 2.57 nm position resolution is expected. Considering measured position resolution calibrated to nominal bunch charge being 5.94 (nm), there are unknown noise equivalent to \( \sqrt{5.94^2 - 2.57^2} \approx 5.35 \) nm other than thermal noise of the electronics. This unknown noise corresponds to 7.86 nm under the condition of actual position resolution measurement operated this time. Please note that thermal noise level of the electronics does not change in relation to beam charge, but the position sensitivity of IP-BPM improves proportionally to beam charge. Therefore position signal equivalent to noise level also improves proportionally to beam charge.

Thermal noise under condition of 30 dB, 20 dB, 10 dB attenuation cases were also measured, which are summarized in Figure 10.21. Solid line shows the expected resolution extrapolated from 30 dB data. Measured resolutions are consistent with the expected line.
Figure 10.16: Schematics of thermal noise measurement
Figure 10.17: Calibration Run under 30dB attenuation

Figure 10.18: Measured rms of residual at short Resolution Runs

Figure 10.19: Results of long Resolution Run under no attenuation
Figure 10.20: Correlation between residual and regression parameters left after analysis

<table>
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<th>Value</th>
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<tr>
<td>Calibration Slope (ch/µm)</td>
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</tr>
<tr>
<td>Stat. Error (ch/µm)</td>
<td>68.49</td>
</tr>
<tr>
<td>Sys. Error (ch/µm)</td>
<td>2.66</td>
</tr>
<tr>
<td>Resolution Run RMS (µm)</td>
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<tr>
<td>Stat. Error (µm)</td>
<td>0.64</td>
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<tr>
<td>Sys. Error (µm)</td>
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<td>Geometry factor</td>
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<tr>
<td>Dynamic Range (µm)</td>
<td>6.57</td>
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<tr>
<td>Position Resolution (nm)</td>
<td>3.86</td>
</tr>
<tr>
<td>Stat. Error (nm)</td>
<td>0.15</td>
</tr>
<tr>
<td>Sys. Error (nm)</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Table 10.6: Results of thermal noise measurement under no attenuation
10.8 Suspection of noise origin

As discussed in the former section, unknown noise equivalent to 7.86 nm position signal was observed during position resolution measurement. There are some suspects which could explain this unknown noise. Those are temeprature vibration, mechanical vibration of the IP-BPM block, and others.

10.8.1 Temperature Vibration

Temperature drift of various points during the 1 hour resolution run is shown in Figure 10.22. The figure includes temperatures of the air inside of the Damping Ring, air inside the plastic hut, sensor and reference cavity surface, and down converter module. Although a 1500 sec cycle structure is seen in DR air temperature, no such structure is seen in the time dependency of residual, shown in Figure 10.8. The cavity surface was stable within 0.01 K, which was actually limited by the resolution of the heat gauge. Considering these results, temperature vibration turned out to be innocent.
10.8.2 Mechanical Vibration

If the whole system, two IP-BPM blocks and the stage which they are installed on, behaves rigidly, mechanical vibration would not contribute to reduce position resolution. However, if each IP-BPM cavity vibrates independently, it can reduce resolution.

Mechanical vibration of the IP-BPM block was checked by using a laser interferometer. A mirrored surface was attached on cavity blocks, and interference between the laser reflecting at the cavity and the reference laser was monitored. Vertical vibrations of the blocks were measured in this way, and its schematics are shown in Figure 10.23. Observed wave forms are shown in Figure 10.24. The frequency and the amplitude of the wave form correspond to those of the vibration of IP-BPM block. Upper wave form represents the down stream BPM block, while the bottom represents the up stream block. Their vibration frequencies are same, both approximately 40 Hz, but their amplitude differs, which implies that the whole IP-BPM system is not a perfect rigid body. The rms of vibration was measured to be within 4.3 nm. Please note that this value includes the rigid vibration, which does not affect the resolution. Therefore, contribution of the mechanical vibration is considered to be smaller than 4.3 nm, and turned out not to be dominant of the unknown noise.

10.8.3 Other Suspects

Considering (4.13), there are several beam originated components which can affect the output signal. Those are, beam charge $q$ and bunch length $\sigma_z$. Although $q$ is monitored by reference cavities, $\sigma_z$ was not monitored in our measurement. Vibration in $\sigma_z$ must be monitored in future beam operation.

Other terms, such as resonant frequency $\omega$, quality factor $Q_{ext}$, $R/Q$ and detecting impedance $Z$ are considered to fluctuate in relation to temperature. Although there was no particular correlation between the measured residual and temperature in order of 10 mK, temperature drift must be monitored with higher resolution.

If the unknown noises originate from the beam, their contributions are expected to change in relation to beam condition, for example beam charge. On the other hand, contribution of unknown noise which doesn’t originate from beam would be constant under various beam condition. Therefore, origin of these unknown noises can be studied by measuring their contribution while changing beam condition.
By determining all noise components and applying them to analysis, principally position resolution of IP-BPM can improve to thermal noise level, which is 2.57 nm as a result of our measurement.
11 Position Resolution Measurement (Multiple Bunch)

At ATF2, not only single bunch operation but multiple bunch operation is also planned. Bunch interval will be ILC like, thus nominal mode 308 ns, and low Q mode 154 ns. Bunch number is planned to be 60.

IP-BPM must be able to measure beam position with nano-meter precision also in multiple bunch modes. Therefore, position resolution in multiple bunch condition was also measured at ATF extraction line. In this section, analysis method and results of multiple operation resolution measurement are discussed in detail.

11.1 Expected Resolution in Multiple Bunch Mode

In case of multiple bunch measurement, output signal would be the superposition of each single bunch signal. If decay time constant $\tau$ of the cavity is much longer than the bunch interval, beam position of each bunch is very difficult to be measured. To measure the beam position bunch to bunch, $\tau$ must be shorter enough compared to the bunch interval.

If the effect of the former bunch is still left, output would be the superposition of multiple bunches, as shown in Figure 11.1. The superposition pattern is determined by the bunch interval $t_b$ and the resonant angular frequency $\omega$ of the cavity. In case of $\omega t_b = 2\pi N$, the signals would sum up. On the other hand in case of $\omega t_b = 2\pi N + \pi$, signals would cancel out the former bunch. Note that $N$ is an integer.

At analyzing, one must separate each bunch by subtracting the contribution from the former bunches. Due to this subtraction of former bunch information, position resolution would be degenerated. For instance, in case of 3 bunches, defining the single bunch resolution $R$ and the decay rate of the 1st and 2nd bunch at the timing of the 3rd bunch as $\alpha$ and $\alpha'$ respectively, the position resolution of the 3rd bunch $R'$ would be

$$R' = \sqrt{1 + \alpha^2 + \alpha'^2 R}. \quad (11.1)$$

If signal decay time is shorter enough than bunch interval, $\alpha$ is negligible and thus resolution $R'$ does not degenerate compared to $R$.

In addition, in case of each bunch summing up, dynamic range can be limited due to saturation of the detecting electronics. The sum of the bunch signals would be

$$V_\infty = \frac{V_0 + V_0 e^{-\frac{t_b}{2\tau}} + V_0 e^{-\frac{2t_b}{2\tau}} + \cdots}{1 - e^{-\frac{t_b}{2\tau}}} = \frac{V_0}{1 - e^{-\frac{2t_b}{2\tau}}} \quad (11.2)$$

In order not to limit the dynamic range, some kind of condition such as $V_\infty < 2V_0$ is needed.

11.2 Bunch Separation and Phase Detection

To check the resolution of IP-BPM, 154 ns bunch interval is enough, since expected resolution is worse than in case of 308 ns. At current ATF extraction line, bunch number is limited to 3 from the length of the Damping Ring and spec of the extraction kicker. Considering decay time of IP-BPM sensor cavity, ~60 ns in amplitude, sensor signal reduces to 0.077 at 154 ns and 0.006 at 308 ns, respectively. However, reference cavity signal does not reduce enough in this bunch interval. Decay time of reference cavity signal is ~360 ns in amplitude, and reduces to 0.65 at 154 ns and 0.43 at 308 ns, respectively.

Please note that decay time of the reference cavity signal is long in order not to saturate the electronics, since common modes are much stronger than di-pole modes. Also, in order to be used as reference signal at analog detection, reference cavity signal is input to limiter detector and output as constant amplitude 714 MHz signal. To generate this 714 MHz signal of constant intensity, sharp decay in input reference cavity signal was avoided.

Assuming that reference cavity contamination is dominant, by applying its decay rate to (11.1), resolution of 2nd bunch and 3rd bunch would be approximately $1.2R$ and $1.3R$, respectively.
Position Resolution Measurement (Multiple Bunch)

Figure 11.1: Multiple bunch signals

\( R \) is the resolution in case of single bunch. Therefore resolution in case of multiple bunches is principally degenerated by 20 ~ 30%.

Also due to these contaminations of multiple bunch signals, mainly two difficulties occur. They are bunch separation and phase detection. Since multiple bunch signals are contaminated, signal phase cannot be detected by analog detection. As discussed in section 7.1 sensor signal and reference signal are detected independently by a common L.O. 714 MHz signal. Then, signal of each bunch consisting the sensor or reference signal are separated digitally, by analysis. Finally, signal of each bunch is detected independently by analytically calculating the phase difference between sensor phase and reference phase of the considering bunch. Thus much more analysis is needed after digitalizing in case of multiple bunch operation, compared with single bunch case.

However, these problems are avoidable if there is a beam synchronized reference signal with no contamination, other than our reference cavity. In this case principally analog phase detection is applicable, and reference cavity is used for monitoring beam charge only. Although bunch separation must be done by analysis, its effect to position resolution is limited since decay rate of sensor cavity signal is under 10%. Therefore, beam synchronized reference signal is absolutely desired for multiple bunch operation.

11.3 Beam Synchronized Phase Origin

To acquire phase origin, beam synchronized reference signal is necessary. The most hopeful candidate is the 714 MHz accelerating RF signal of the Damping Ring, since beam bunch is accelerated at a particular region of the RF phase. However, it was proved that its phase stability was not enough to be used for phase detection, by the following measurement.

To check the phase stability between DR 714 MHz and beam timing, correlation between DR 714 MHz and our reference cavity signal was monitored. DR 714 MHz is picked up from the DR and is transmitted to "RF Hut", locating just by the DR gate, outside of the DR. After passing simple electronics and amplifiers, it is transmitted to the "eel’s bedroom", via Damping Ring. At the eel’s bedroom, reference cavity signal is detected by the DR 714 MHz. Both signals are transmitted to eel’s bedroom from DR using same cables. The schematics of this measurement are shown in Figure 11.2.
From I signal and Q signal, phase difference of DR 714 MHz and reference signal is observed. It was measured for about 2 hours. The results are shown in Figure 11.3. The left figure is the I-Q plot, and pedestal is shown in red. As seen in the figure, approximately 1 rad phase drift is observed. The right figure is the time dependency of the detected phase, and 10 minutes cycle structure is observed.

Please note that this DR 714 MHz signal is the one which is used to generate 5.712 GHz L.O. signal for first down conversion. In case of first down conversion, phase drift between beam timing is not a problem since it is balanced out, as shown in (7.10).

As shown in (7.9), the reference signal is represented as

\[ Y_{ref} = \sin(\omega_0 t + (\phi_{ref} - 8\phi_{LO})) \]  

(11.3)

Since it is detected by the same L.O., DR 714 MHz, which is represented as

\[ Y_{LO} = \sin(\omega_0 t + \phi_{LO}) \]  

(11.4)
the detected phase is

\[ \sin(\phi_{\text{ref}} - 9\phi_{\text{LO}}). \] (11.5)

Therefore, this observed 1 rad is the phase drift against C band signal. Genuine phase drift in the ATF DR 714 MHz is approximately 1/9 rad, which is \( \sim 6.4 \) degrees.

Intensity of DR 714 MHz and reference cavity signal were also monitored, but there was no correlation to be seen between detected phase and those intensities. Therefore, the origin of this phase drift is not our electronics. However, beam timing cannot drift in principle, since it cannot be accelerated in improper timing. As a result, the most suspicious is the cable length, which can possibly change in relation to temperature drift.

Considering 714 MHz signal, 1 cycle is \( \sim 1.4 \) ns. Therefore, 1/9 rad phase drift correspond to \( \sim 5 \) mm change in cable length, since light speed \( c \) in coaxial cable is 5 ns/m. The cable used for transmitting DR 714 MHz signal from RF Hut to DR was 50 m, and thus \( 10^{-4} \) length change can cause such phase drift.

Temperature drift of 10 minutes cycle was observed in the RF Hut, where the DR 714 MHz cable goes through. The temperature of the hut is controlled by an air conditioner, which simply turns on at a particular temperature to cool the room down, and turns off at another particular temperature. This air conditioning system is the cause of the zigzag pattern observed as phase drift. Especially near the fan, temperature change was approximately 20 degrees, and the DR 714 MHz cable was passing just near by.

When temperature of a coaxial cable changes, mainly two effects occur that can affect signal phase. First, actual cable length changes due to thermal expansion of the metal and insulator. Second, insulator permittivity changes, which affects the effective speed of the signal transmitting through the cable. In case of temperature drift, the second effect is dominant.

However, phase drift did not change even when the cable was covered with heat insulating material. The material’s effect was confirmed before the measurement to reduce temperature drift to approximately 1/3, and it must be observed if the phase drift was due to the cable. As a result, cable was also innocent.

So far, we suspect the band pass filter at RF Hut, shown in Figure 11.2, to be the origin of the phase drift. Band pass filters consist of high pass filters and low pass filters, which can possibly cause phase shift under temperature variation.

We have failed to acquire beam synchronized reference signal anyway, and decided to use this DR 714 MHz as L.O. signal to detect sensor and reference signals at the eel. However, solving this problem by giving feedback or analytical correction is strongly desired for future operation.

### 11.4 Analysis Method

As discussed in advance, the key points of multiple bunch measurement are bunch separation and digital phase detection. Since phase detection is described in section 7.1, details of bunch separation are described in this section.

The measurement scheme is shown in Figure 11.4. Detecting scheme is the same as shown in Figure 7.8. However in addition, in order to detect 3 bunches at different timing, 3 charge ADCs are used for 1 sensor signal. Defining the first bunch timing as \( t = 0 \), each ADC gate timing is set to \( t = 0, 154 \) ns, and 308 ns, respectively.

To subtract the effect of former bunch, it is necessary to understand the decay rate \( \alpha \) and phase transition \( \theta \) of the former bunch. Let’s define measured I vector and Q vector of \( n \) th bunch as \( I_n \) and \( Q_n \). At \( n \) th bunch timing, \( n - 1 \) th bunch signal decays by \( \alpha \) and rotates \( \theta \), as shown in Figure 11.5.

Phase transition occurs since the sensor signal frequency and reference signal frequency are not exactly the same. Frequency difference between sensor signals or reference signals and the DR 714 MHz, \( \Delta f \), is approximately 3 MHz. In case of \( \Delta f = 3 \) MHz and bunch interval \( t_b = 154 \) ns, phase transition during bunch interval is \( \Delta \omega t_b \simeq 165.6^\circ \). This frequency difference is inevitable since all 3 sensor signals have to be detected by a common L.O. signal. Unless all 3 sensor signal frequencies equate exactly, frequency difference occurs.

In order to measure decay rate \( \alpha \) and phase transition \( \theta \), single bunch operation under multiple bunch detecting scheme is carried out. In this case, only the single bunch is measured at each
Figure 11.4: Schematics of multiple bunch measurement

Figure 11.5: Contamination of former bunch in I-Q surface
11 POSITION RESOLUTION MEASUREMENT (MULTIPLE BUNCH)

timing, and therefore bunch separation is not necessary. From analysis, one can determine signal amplitude and phase at each timing. From these results, $\alpha$ and $\theta$ can be determined.

Once they are determined, genuine $n$th bunch signal $I_{ngen}$ and $Q_{ngen}$ are obtained as

$$
\begin{bmatrix}
I_{ngen} \\
Q_{ngen}
\end{bmatrix}
= \begin{bmatrix}
I_n \\
Q_n
\end{bmatrix} - \alpha \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
I_{n-1} \\
Q_{n-1}
\end{bmatrix}.
$$

(11.6)

This method can be applied at any bunch number.

Once bunch separation is applied, phase detection can be possible. As shown in section 7.1, sensor signal amplitude $A_{sen}$, reference signal amplitude $A_{ref}$, and sensor signal phase $\phi = \phi_{sen} - \phi_{ref}$ are obtained from analysis. Therefore,

$$
I_{sen} = A_{sen} \times \cos \phi
$$

$$
Q_{sen} = A_{sen} \times \sin \phi
$$

(11.7)

are defined. In case of unlocked L.O., I-Q tuning cannot be done since L.O. phase shifts randomly. However in case of DR 714 MHz, phase drift is $\sim 1$ rad, which enables to tune position signal to $I$ and others to $Q$, although fine tuning is not applicable. By choosing the component sensitive to beam position, for example $I$, Calibration Run can be applied to measure its relation to beam position. At Resolution Run, linear regression analysis is applied using the following parameters:

$$
Y_{2I_p} = \alpha_0 + \alpha_{Y_{1I}} Y_{1I} + \alpha_{Y_{1Q}} Y_{1Q} + \alpha_{Y_{3I}} Y_{3I} + \alpha_{Y_{3Q}} Y_{3Q} + \alpha_{Y_{ref}} A_{ref},
$$

(11.8)

where $Y_{1I}$, $Y_{1Q}$ are $I_{sen}$ and $Q_{sen}$ signals of BPM1, and $Y_{3I}$, $Y_{3Q}$ are $I_{sen}$ and $Q_{sen}$ signals of BPM3. X components are not included since they could not be taken in 3 bunch timings due to limitation of charge ADC channels and number of divider modules. X component signals, such as $X_{1I}$, $X_{1Q}$, $X_{3I}$, $X_{3Q}$, and XREF were acquired at 1st bunch timing only, using analog detection. Though they were not used for regression analysis, XREF was used for beam charge cut, since YREF amplitude cannot be monitored by analog detection.

11.5 Measurement Procedure

The measurement procedure is a little bit complicated compared to single bunch operation, as summarized in the following.

1. Reference cavity calibration with single bunch mode, in order to calibrate the reference cavity signal to beam charge.
2. I-Q tuning, which is not necessary in case of absolutely unlocked L.O..
3. Pedestal Run, to determine the origin of I-Q surface.
4. Calibration Run with single bunch mode, in order to measure amplitude decay and phase transition of sensor signal.
5. Resolution Run with single bunch mode, to check the single bunch resolution if needed.
6. Calibration Run with multiple bunch mode.
7. Resolution Run with multiple bunch mode.

Basically the analysis method is same with single bunch case. However, Calibration Run with single bunch mode is necessary for bunch separation. After bunch information is separated, digital phase detection is applied. Then, the same method as single bunch case is applied to each bunch information. Therefore, beam position resolution of each bunch is obtained independently.

At current ATF beam line, beam orbit of the 3 bunches differ in order of 100 $\mu$m. However, dynamic range of IP-BPM electronics is not that wide without attenuation. In order to measure all 3 bunches within dynamic range of IP-BPM, 40 dB attenuation was applied for both calibration run and resolution run. Bunch to bunch feedback is strongly desired to improve resolution, to apply lower attenuation for beam position measurements.
11.6 Reference Cavity Calibration

Reference cavity calibration method is the same with single bunch operation, and is carried out with single bunch. However, in order to apply ICT cut to data during resolution measurement, Y reference cavity signal cannot be used since its amplitude is calculated by analysis. In this case, X reference cavity signal was used to monitor beam charge. The result is shown in Figure 11.6.

11.7 Pedestal Run

In order to determine signal amplitude and phase, center of I-Q surface must be determined. Actually it corresponds to pedestal of the phase detector, which is the noise level without sensor signal input. Therefore sensor signal input was terminated by 50 Ω, while reference signal from the limiter detector was input. Approximately 200 events were taken for every ADC channels, and their means were determined to be the I-Q surface origins.

11.8 Calibration Run with single bunch

Method of calibration run is just the same as single bunch operation. However in this case, digital phase detection is applied in 3 timings, with 154 ns interval each. Therefore, we can understand how single bunch decays at 2nd bunch timing and 3rd bunch timing. Applying \((7.14)\) and \((7.15)\), signal amplitude \(A(t)\) and phase \(\theta(t)\) at each timing are determined. By comparing these amplitude and phase among the 3 timings, decay rate \(\alpha\) and phase transition \(\Delta\theta\) are determined as

\[
\alpha = \frac{A(t_{n-1})}{A(t_n)},
\]

\[
\Delta\theta = \theta(t_n) - \theta(t_{n-1}),
\]

where \(t_n\) is the \(n\) th bunch timing.

Time evolutions of single bunch signals in I-Q surface are shown in Figure 11.7. As shown in these figures, signal amplitude, which is represented as the distance from origin, is decreasing. Phase transition is represented as rotation in the I-Q surface.

As a result of calibration run, signal amplitude forms a V shape in relation to beam position, which is shown in Figure 11.8, 11.9, and 11.10. By comparing the slope of V shape between 1st bunch timing and 2nd bunch timing, \(\alpha\) is obtained. Also, mean of phase transition at each fixed beam position of the calibration run was considered to be \(\Delta\theta\). Obtained signal decay rate \(\alpha\) and phase transition \(\Delta\theta\) are summarized in Table 11.1. Phase transition is consistent with the frequency difference between sensor signals and reference 714 MHz. However, comparing \(\alpha\) with
Figure 11.7: Time evolution of sensor and reference signal in I-Q surface

<table>
<thead>
<tr>
<th></th>
<th>Amplitude at t = 0 (count/µm)</th>
<th>Amplitude at t = 154 ns (count/µm)</th>
<th>Amplitude at t = 308 ns (count/µm)</th>
<th>Signal Decay $\alpha$</th>
<th>Phase Transition $\Delta \theta$ (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPM1</td>
<td>4.85</td>
<td>1.17</td>
<td>0.19</td>
<td>0.24</td>
<td>3.23</td>
</tr>
<tr>
<td>BPM2</td>
<td>6.15</td>
<td>1.95</td>
<td>0.25</td>
<td>0.32</td>
<td>3.32</td>
</tr>
<tr>
<td>BPM3</td>
<td>5.07</td>
<td>1.18</td>
<td>0.21</td>
<td>0.23</td>
<td>3.42</td>
</tr>
<tr>
<td>REF</td>
<td></td>
<td></td>
<td></td>
<td>0.80</td>
<td>3.45</td>
</tr>
</tbody>
</table>

Table 11.1: Decay rate and phase transition during bunch interval

Theoretical prediction, measured signal amplitude decay is slower. Applying decay time constant $\tau$ from Table 6.2 to exponential decay, $\alpha$ must be 0.08, 0.14, 0.06 for BPM1, BPM2 and BPM3, respectively.

Bunch separation (11.6) is assuming an exponential decay. If signal decays exponentially, decay rate $\alpha$ would be constant if considering time interval is constant, which makes it possible to consider only the adjacent bunch. In this case, same algorithm can be applied in case of any bunch number, since there is no need to consider all other former bunches. However, considering the decay rate between $t = 154$ ns and $308$ ns, decay rate is not the same with that of $t = 0$ ns and $154$ ns. Fortunately, decay rate of sensor signals at $308$ ns interval is approximately 0.04, and their contributions are negligible.
11.9 Calibration Run

Next, calibration run for 3 bunch mode is carried out. Applying $\alpha$ and $\Delta\theta$ determined from single bunch mode calibration run, I signal of each bunch can be calibrated to actual beam displacement independently.

Measured amplitude of 1st bunch, 2nd bunch, and 3rd bunch at each cavity are shown in Figure 11.8, 11.9, and 11.10. If bunch separation has succeeded, slopes of the amplitude must be similar at all 3 beam timings. From these figures, bunch separation seems to be working.

Results of calibration run are shown in Figure 11.11, 11.12, and 11.13. The vertical errors shown in the figures are the statistical errors of I signal and beam position measured by stripline BPMs. The slopes are summarized in Table 11.2. If bunch separation has succeeded, these slopes must be similar at all 3 bunch timings. From Table 11.2, the slope of 2nd bunch is approximately 10% larger compared with 1st bunch and 3rd bunch, but it seems to be working basically.

11.10 Resolution Run

At a fixed beam orbit near cavity center, resolution run was taken for approximately 1000 events. The analysis method is the same with single bunch case, after separating bunch information.
Figure 11.11: Signal amplitude of 3 bunches measured at BPM1

Figure 11.12: Signal amplitude of 3 bunches measured at BPM2

Figure 11.13: Signal amplitude of 3 bunches measured at BPM3

Figure 11.14: I signal of 3 bunches measured at BPM1

Figure 11.15: I signal of 3 bunches measured at BPM2

Figure 11.16: I signal of 3 bunches measured at BPM3
Table 11.2: Measured calibration slope at each cavities

Linear regression analysis was applied, and from RMS of the residual

\[ Y_{2I_{\text{measured}}} - Y_{2I_{\text{predicted}}} \]  

and geometrical factor, position resolution of each bunch is obtained.

Figure 11.17, 11.19 and 11.21 show the residual of 1st bunch, 2nd bunch and 3rd bunch, respectively. Measured \( Y_{2I} \) is plotted against predicted \( Y_{2I} \) in the left figures, which width corresponds to the residual between them. Right figures show the distribution of the residual, and their RMS correspond to the resolution of the measurement.

Figure 11.18, 11.20 and 11.22 show the correlation between residual and regression parameters left after analysis. As seen in the figures, no particular correlations are seen to be left.

If bunch separation has succeeded, there must be no correlation between the residual and former bunch information. To confirm this fact, correlation between residual of 2nd bunch and regression parameters of 1st bunch is shown in Figure 11.23. Also, correlation between residual of 3rd bunch and regression parameters of 2nd bunch is shown in Figure 11.24. As shown in the figures, no particular correlations are seen.

The position resolution of each bunch is calculated from

\[ (10.5) \]

As shown in the table, sub-micron resolution is proved under 40 dB attenuation. Statistic errors are calculated from \( \text{RMS of residual}/\sqrt{2N} \), where \( N \) is the total event number. Systematic errors are the errors from the calibration slope. Resolutions extrapolated to no attenuation and nominal charge are also shown. Simply extrapolating, nano-meter resolution is achievable.

Beam charge (ICT) of each bunch was estimated from the measured \( Y \) reference cavity signal amplitude. Although \( X_{\text{REF}} \), the amplitude of \( X \) reference cavity signal used for ICT cut, was monitored at 1st bunch timing only, \( Y_{\text{REF}} \) can be calibrated to \( X_{\text{REF}} \) by measuring their correlation during measurement. Since both \( X_{\text{REF}} \) and \( Y_{\text{REF}} \) are sensitive to beam charge, they have strong correlation between each other, as shown in Figure 11.25. From Figure 11.25 and Figure 11.6, \( Y_{\text{REF}} \) signal of each bunch is calibrated to beam charge. As shown in Table 11.3, beam charge of the 3rd bunch is lower compared with other bunches.

Considering the discussion in former sections, resolution of 2nd bunch and 3rd bunch are expected to be 20% ~ 30% worse compared with the 1st bunch. However, measured ones are much worse than expected. This is considered to be originated from the bunch separation method, applying not only amplitude decay but also phase transition. Also as shown in Figure 11.26, beam jitter of 2nd bunch and 3rd bunch were larger compared with the 1st bunch.

Beam distribution during the resolution run is shown in Figure 11.26 which shows the distribution of measured \( Y_{2I} \), the position signal at BPM2. Although calibration factor of \( Y_{2I} \) signal of each bunch to actual beam position differ, approximately 8 counts of horizontal correspond to 1 \( \mu \)m. Therefore, beam jitter during the operation was approximately 20 \( \mu \)m, and beam orbit difference among the 3 bunches is approximately 300 \( \mu \)m. This orbit difference is considered to be the effect of incomplete flat-top wave form of the extraction kicker.
Figure 11.17: Distribution of the residual of 1st bunch

Figure 11.18: Correlation between residual and parameters left for 1st bunch
Figure 11.19: Distribution of the residual of 2nd bunch

Figure 11.20: Correlation between residual and parameters left for 2nd bunch
Figure 11.21: Distribution of the residual of 3rd bunch

Figure 11.22: Correlation between residual and parameters left for 3rd bunch
Figure 11.23: Correlation between residual of 2nd bunch and parameters of 1st bunch
Figure 11.24: Correlation between residual of 3rd bunch and parameters of 2nd bunch
Table 11.3: Measured position resolution under 3 bunch operation

<table>
<thead>
<tr>
<th>Bunch</th>
<th>Position Resolution (nm)</th>
<th>Stat. Error (nm)</th>
<th>Sys. Error (nm)</th>
<th>ICT during measurement (×1.6 nC)</th>
<th>Resolution extrapolated to nominal charge without attenuation (nm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st bunch</td>
<td>403.3</td>
<td>8.29</td>
<td>9.50</td>
<td>0.56</td>
<td>2.28</td>
</tr>
<tr>
<td>2nd bunch</td>
<td>798.9</td>
<td>16.4</td>
<td>22.9</td>
<td>0.57</td>
<td>4.58</td>
</tr>
<tr>
<td>3rd bunch</td>
<td>812.6</td>
<td>16.7</td>
<td>25.8</td>
<td>0.51</td>
<td>4.15</td>
</tr>
</tbody>
</table>

Figure 11.25: Correlation between YREF signal and XREF signal

Figure 11.26: Bunch distribution during the resolution run
12 Conclusion

An ultra high resolution cavity BPM, which is called the IP-BPM, is developed for final focus system of ATF2. In order to monitor beam position with nano-meter resolution at ATF2 IP, IP-BPM is designed to have ultra high position sensitivity. Beam tests at ATF extraction line proved that IP-BPM has sufficient sensitivity to detect 2 nm position signal in vertical. Also, angle sensitivity of IP-BPM was proved to be reduced enough, in order to monitor beam position free from large angle jitter at ATF2 IP.

Position resolution of IP-BPM under single bunch operation was proved to be $8.72 \pm 0.28 \pm 0.35$ nm in vertical, with $5 \mu m$ dynamic range, stable for 1 hour. However in order to meet ATF2 requirements, resolution must be improved to thermal noise level, which is measured to be 2.6 nm in nominal ATF2 beam charge. Therefore, determination and correction of unknown noise other than thermal noise are strongly desired. Principally, IP-BPM can achieve thermal noise level resolution.

Multiple bunch operation is also planned at ATF2, and therefore IP-BPM must be able to monitor beam position of each bunch independently, with nano-meter resolution. However due to narrow dynamic range of IP-BPM, it is difficult to demonstrate nano-meter resolution at current ATF beam line. Introducing 40 dB attenuation to the output position signal, IP-BPM succeeded to measure beam position of each bunch independently with sub-micron resolution. In order to improve resolution, beam orbit fluctuation among bunches must be controlled. Also, beam synchronized reference signal is desired in order to reduce contamination from former bunches.

ATF2 will start its operation in autumn of 2008, and at first IP-BPM will be installed with Shin-take Monitor, to demonstrate 37 nm beam size. At phase II, IP-BPM will be used to demonstrate beam orbit control with nano-meter stability at ATF2 IP.
13 Acknowledgements

Firstly, I would gratefully appreciate my supervisor, Professor Sachio Komamiya, for his continuous encouraging me with lots of useful advices about every part of my study. He was the best professor I could ever learn under, and provided me lots of chances and kindness to study physics, go through experiments, make presentations, and so on.

I would also appreciate the former assistant professor of our laboratory, Professor Tomoyuki Sanuki, for his great advices for my work. It was him who provided me with this theme for my master course. Also, I would be thankful for the current assistant professor Dr. Yoshio Kamiya for many advices and kindness.

I would also gratefully acknowledge my supervisor at ATF, Dr. Yosuke Honda, with deepest appreciation for his gracious and dedicated supervising on experiments and KEK life. He was the best person to learn experimental techniques from and provided me lots of chances to go through treasurable experiments and bring valuable results. Also I sincerely appreciate my co-workers of the Cavity BPM group, Mr. Yoichi Inoue and Dr. Seunghwan Shin, for their helpful support. I would also be thankful for Mr. Toshikazu Takatomi of Mechanical Engineering Center of KEK. His precise processing technique enabled us to fabricate IP-BPM with such high accuracy.

I would also appreciate Professor Junji Urakawa, Dr. Nobuhiro Terunuma, and many other members of ATF collaboration for the financial support and their great efforts of operating and maintaining the ATF, and constructing ATF2. Also I am deeply grateful for Mr. Sakae Araki, Dr. Hiroshi Sakai, Dr. Masahumi Fukuda, Mr. Ken Watanabe, Mr. Kazuyuki Sakaue and all other habitants of the "Second Contenna", for their kind physical and mental support for my KEK life, including food, dishes, transportations and so on. I also appreciate Dr. Yasutika Yamamoto for lending me bicycle at KEK, and for his unique advices and encouragements. I also express my appreciation to Ms. Sasaki and Ms. Ikeda, for taking care of office procedure of my KEK life. I could really enjoy my life as a scientist at ATF.

I would also be grateful for all other members of our laboratory, Dr. Taiki Yamamura, Mr. Taikan Suehara, Ms. Yasuko "Peko" Hisamatsu, Mr. Hideki Okawa, Mr. Shinzuke Kawasaki, Mr. Shinya Sonoda, Master Hakutaro Yoda, Mr. Takashi Ymanaka and Mr. Masahiro Oroku, for helpful support for my life in the laboratory. I also appreciate Ms. Kono and all the members of ICEPP secretary’s office, for their kind and continuous support.

Lastly I appreciate my family, Yoichi, Akiko, Yuka and Satoshi Nakamura, for continuous financial support and encouragement to study physics.

I am very proud to have had chances to join ATF2 and ILC project during my master course. I believe not before long, ILC would reveal the truth of nature, and technologies developed at ATF2 would gratefully contribute this promised historic discovery.
A APPENDIX

A.1 Electromagnetic field of cavities

A.1.1 Starting point – Maxwell Equation

The electromagnetic field of a cavity is deductively calculated from Maxwell Equation. Maxwell Equation is represented as below:

\[ \nabla \times \mathbf{E} + \mu_0 \frac{\partial \mathbf{H}}{\partial t} = 0 \quad (A.1) \]
\[ \nabla \cdot \mathbf{H} = 0 \quad (A.2) \]
\[ \nabla \times \mathbf{H} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} \quad (A.3) \]
\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}. \quad (A.4) \]

Assuming current \( \mathbf{J} = 0 \) and radio wave transmitting in z direction,

\[ \mathbf{E} \propto e^{i(\omega t + k_z z)} \quad (A.5) \]
\[ \mathbf{H} \propto e^{i(\omega t + k_z z)} \quad (A.6) \]

are obtained. In case of cylindrical or rectangular cavity, electromagnetic field of the cavity can be solved analytically. Otherwise numeric calculation or simulation is needed. ATF2 Q-BPMs and IP-BPM reference cavities are cylindrical, and IP-BPM sensor cavities are rectangular.

A.1.2 Rectangular Cavity

First, let’s consider a rectangular cavity. The cavity length is a, b, L in X, Y, Z direction, respectively. Applying orthogonal coordinate system to (A.1) and (A.3),

\[ \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu_0 \frac{\partial H_x}{\partial t} \quad (A.7) \]
\[ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu_0 \frac{\partial H_y}{\partial t} \quad (A.8) \]
\[ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu_0 \frac{\partial H_z}{\partial t} \quad (A.9) \]
\[ \frac{\partial H_x}{\partial y} = \frac{\partial H_y}{\partial z} = \epsilon_0 \frac{\partial E_z}{\partial t} \quad (A.10) \]
\[ \frac{\partial H_x}{\partial z} = \frac{\partial H_z}{\partial x} = \epsilon_0 \frac{\partial E_y}{\partial t} \quad (A.11) \]
\[ \frac{\partial H_y}{\partial x} = \frac{\partial H_z}{\partial y} = \epsilon_0 \frac{\partial E_x}{\partial t} \quad (A.12) \]

are obtained. Applying (A.5), (A.6) and assuming TM mode \( (H_z=0) \) to (A.7) ~ (A.12),

\[ \frac{\partial E_z}{\partial y} \pm ik_z E_y = -i\omega \mu_0 H_x \quad (A.13) \]
\[ \mp ik_z E_x - \frac{\partial E_z}{\partial x} = -i\omega \mu_0 H_y \quad (A.14) \]
\[ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \quad (A.15) \]
\[ \pm i k_z H_y = i \omega \epsilon_0 E_x \quad (A.16) \]
\[ \mp i k_z H_x = i \omega \epsilon_0 E_y \quad (A.17) \]
\[ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = i \omega \epsilon_0 E_z \quad (A.18) \]

are obtained. Applying (A.14) to (A.16),
\[ \frac{\partial E_z}{\partial x} = i \frac{k^2 - k_z^2}{\omega \epsilon_0} H_y = \pm i \frac{k^2 - k_z^2}{k_z} E_x \quad (A.19) \]
is obtained. Likewise, by applying (A.13) to (A.17),
\[ \frac{\partial E_z}{\partial y} = -i \frac{k^2 - k_z^2}{\omega \epsilon_0} H_x = \pm i \frac{k^2 - k_z^2}{k_z} E_y \quad (A.20) \]
is obtained. Note that the following,
\[ \epsilon_0 \mu_0 = 1/c^2, \, k \equiv \omega/c \quad (A.21) \]
is used. Applying transformation of (A.19) and (A.20),
\[ H_y = -i \frac{\omega \epsilon_0}{k^2 - k_z^2} \frac{\partial E_z}{\partial x} \quad (A.22) \]
\[ H_x = i \frac{\omega \epsilon_0}{k^2 - k_z^2} \frac{\partial E_z}{\partial y} \quad (A.23) \]
to (A.18) and then ordering,
\[ \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + (k^2 - k_z^2) E_z = 0 \quad (A.24) \]
is obtained. Applying variable separation of \( E_z \)
\[ E_z = X(x)Y(y)e^{i(\omega t + k_z z)} \quad (A.25) \]
to (A.24) and then ordering,
\[ \frac{1}{X} \frac{d^2 X}{dx^2} + (k^2 - k_z^2) = -\frac{1}{Y} \frac{d^2 Y}{dy^2} \quad (A.26) \]
is obtained. Since left-hand member is a function of \( x \) only, while the right hand member is a function of \( y \) only, this value must be a constant. Representing the constant as \( k_y^2 \), (A.26) is written as
\[ \frac{1}{Y} \frac{d^2 Y}{dy^2} = -k_y^2 \]
\[ \longrightarrow Y = C \cos k_y y + D \sin k_y y. \quad (A.27) \]
Likewise, by using a constant \( k_x \)
\[ \frac{1}{X} \frac{d^2 X}{dx^2} = -k_x^2 \]
\[ \longrightarrow X = A \cos k_x x + B \sin k_x x \quad (A.28) \]
is obtained. Using these terms, (A.29) can be written as

$$E_z = (A \cos k_x x + B \sin k_x x)(C \cos k_y y + D \sin k_y y) e^{i(\omega t + k_z z)}. \quad (A.29)$$

Notice that

$$k_x^2 + k_y^2 + k_z^2 = k^2 \quad (A.30)$$

is formed. Applying (A.29) to (A.19) and (A.20),

$$E_x = \mp i \frac{k_x}{k^2 - k_y^2} \frac{\partial E_x}{\partial x}$$

$$E_y = \mp i \frac{k_y}{k^2 - k_x^2} \frac{\partial E_y}{\partial y}$$

are obtained.

Boundary condition of a condutor surface requires that electric field parallel to the surface $E_{//}$ and magnetic field transverse to the surface $H_{\perp}$ to be zero. This is because $E_{//}$ is continuous at the surface, while there is no electric field inside a conductor. Therefore it is represented as

$$E_y = E_z = 0 \quad \text{at} \quad x = 0, a$$

$$E_x = E_z = 0 \quad \text{at} \quad y = 0, b.$$ \quad (A.33) (A.34)

This leads to

$$A = 0, \quad \sin(k_y a) = 0, \quad k_x = \frac{m\pi}{a} \quad (m : \text{integer}) \quad (A.35)$$

$$C = 0, \quad \sin(k_y b) = 0, \quad k_y = \frac{n\pi}{b} \quad (n : \text{integer}). \quad (A.36)$$

Thus electric field is calculated as

$$E_z = E_0 \sin \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) e^{i(\omega t + k_z z)}. \quad (A.37)$$

$$E_x = \mp i \frac{k_x}{k^2 - k_y^2} E_0 \frac{m\pi}{a} \cos \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) e^{i(\omega t + k_z z)}. \quad (A.38)$$

$$E_y = \mp i \frac{k_y}{k^2 - k_x^2} E_0 \frac{n\pi}{b} \sin \left( \frac{m\pi}{a} x \right) \cos \left( \frac{n\pi}{b} y \right) e^{i(\omega t + k_z z)}. \quad (A.39)$$

The actual field is the superposition of electromagnetic wave transmitting to plus and minus direction in $z$, and therefore they are written as

$$E_z = E_0 \sin \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) (A_- e^{-ik_z z} + A_+ e^{ik_z z}) e^{i\omega t}. \quad (A.40)$$

$$E_x = i \frac{k_x}{k^2 - k_y^2} E_0 \frac{m\pi}{a} \cos \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) (A_- e^{-ik_z z} + A_+ e^{ik_z z}) e^{i\omega t}. \quad (A.41)$$

$$E_y = i \frac{k_y}{k^2 - k_x^2} E_0 \frac{n\pi}{b} \sin \left( \frac{m\pi}{a} x \right) \cos \left( \frac{n\pi}{b} y \right) (A_- e^{-ik_z z} + A_+ e^{ik_z z}) e^{i\omega t}. \quad (A.42)$$
Since boundary condition is given as
\[ E_x = E_y = 0 \text{ at } z = 0, L, \]  
from (A.41) and (A.42)

\[ E_x(z = 0) = E_y(z = 0) = 0 \quad \longleftrightarrow \quad A_- = A_+ \]  
\[ E_x(z = L) = E_y(z = L) = 0 \quad \longleftrightarrow \quad k_z = \frac{l\pi}{L} \]

are obtained. Finally, TM_{mnl} mode of a rectangular cavity is represented as

\[ E_z = E_0 \sin \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) \cos \left( \frac{m\pi}{a} x \right) e^{i\omega t} \]  
\[ E_x = -\frac{k_z}{k^2 - k_z^2} E_0 \frac{m\pi}{a} \cos \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) \sin \left( \frac{l\pi}{L} z \right) e^{i\omega t} \]  
\[ E_y = -\frac{k_z}{k^2 - k_z^2} E_0 \frac{n\pi}{b} \sin \left( \frac{m\pi}{a} x \right) \cos \left( \frac{n\pi}{b} y \right) \sin \left( \frac{l\pi}{L} z \right) e^{i\omega t} \]  
\[ H_z = 0 \]  
\[ H_x = i \frac{\omega \varepsilon_0}{k^2 - k_z^2} E_0 \frac{n\pi}{b} \sin \left( \frac{m\pi}{a} x \right) \cos \left( \frac{n\pi}{b} y \right) \cos \left( \frac{l\pi}{L} z \right) e^{i\omega t} \]  
\[ H_y = -i \frac{\omega \varepsilon_0}{k^2 - k_z^2} E_0 \frac{m\pi}{a} \cos \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) \cos \left( \frac{l\pi}{L} z \right) e^{i\omega t}. \]

It is easily shown that \( E_{//} \) and \( H_{//} \) are zero at cavity surface, which are the requirements from the boundary condition.

Also, by using \( \omega = 2\pi f = c k \), resonant frequency is represented as

\[ f = \frac{1}{2\pi} c \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{c}{2\pi} \sqrt{\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 + \left( \frac{l\pi}{L} \right)^2}. \]

### A.1.3 Cylindrical Cavity

Next, let’s consider a cylindrical cavity. Its diameter is \( a \) and the length in axial direction is \( L \). Applying cylindrical coordinate system to (A.1) and (A.3),

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r E_z \right) - \frac{\partial E_\phi}{\partial z} = -\mu_0 \frac{\partial H_r}{\partial t} \\
\frac{1}{r} \frac{\partial}{\partial r} \left( r E_\phi \right) - \frac{\partial E_z}{\partial r} = -\mu_0 \frac{\partial H_\phi}{\partial t} \\
\frac{1}{r} \frac{\partial}{\partial r} \left( r H_z \right) - \frac{\partial H_\phi}{\partial r} = \varepsilon_0 \frac{\partial E_r}{\partial t} \\
\frac{1}{r} \frac{\partial}{\partial r} \left( r H_\phi \right) - \frac{\partial H_z}{\partial r} = \varepsilon_0 \frac{\partial E_\phi}{\partial t} \\
\frac{1}{r} \frac{\partial}{\partial r} \left( r H_r \right) - \frac{\partial H_z}{\partial \phi} = \varepsilon_0 \frac{\partial E_z}{\partial t} \\
\frac{1}{r} \frac{\partial}{\partial r} \left( r H_\phi \right) - \frac{\partial H_r}{\partial \phi} = \varepsilon_0 \frac{\partial E_z}{\partial t}
\]

are obtained. Please note that rotation expression of cylindrical coordinate

\[ \nabla \times \mathbf{A} = \left( \frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \mathbf{e}_r + \left( \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \mathbf{e}_\phi + \left( \frac{1}{r} \frac{\partial (r A_\phi)}{\partial r} - \frac{1}{r} \frac{\partial A_r}{\partial \phi} \right) \mathbf{e}_z \]
was applied. Applying (A.5), (A.6) and assuming TM mode \((H_z = 0)\) to (A.53) \(\sim\) (A.58),
\[
\frac{1}{r} \partial E_z \frac{\partial}{\partial \phi} \pm ik_z E_\phi = -i \omega \mu_0 H_r
\]
(A.60)
\[
\mp ik_z E_r - \frac{1}{r} \frac{\partial E_z}{\partial r} = -i \omega \mu_0 H_\phi
\]
(A.61)
\[
\frac{1}{r} \frac{\partial (r E_\phi)}{\partial r} - \frac{1}{r} \frac{\partial E_r}{\partial \phi} = 0
\]
(A.62)
\[
\pm ik_z H_\phi = i \omega \epsilon_0 E_r
\]
(A.63)
\[
\mp ik_z H_r = i \omega \epsilon_0 E_\phi
\]
(A.64)
\[
\frac{1}{r} \frac{\partial (r H_\phi)}{\partial r} - \frac{1}{r} \frac{\partial H_r}{\partial \phi} = i \omega \epsilon_0 E_z
\]
(A.65)
are obtained. Applying (A.61) to (A.63),
\[
\partial E_z = i \frac{k^2 - k_z^2}{\omega \epsilon_0} H_\phi = \pm i \frac{k^2 - k_z^2}{k_z} E_r
\]
(A.66)
is obtained. Likewise, by applying (A.60) to (A.64),
\[
\partial E_z = -ir \frac{k^2 - k_z^2}{\omega \epsilon_0} H_r = \pm ir \frac{k^2 - k_z^2}{k_z} E_\phi
\]
(A.67)
is obtained. Applying transformation of (A.66) and (A.67),
\[
H_\phi = -i \frac{\omega \epsilon_0}{k^2 - k_z^2} \frac{\partial E_z}{\partial r}
\]
(A.68)
\[
H_r = \frac{1}{i r} \frac{\omega \epsilon_0}{k^2 - k_z^2} \frac{\partial E_z}{\partial \phi}
\]
(A.69)
to (A.65) and then ordering,
\[
\frac{1}{r} \frac{\partial E_z}{\partial r} + \frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \phi^2} + (k^2 - k_z^2) E_z = 0
\]
(A.70)
is obtained. Applying variable separation of \(E_z\)
\[
E_z = R(r) \Phi(\phi) e^{i(\omega t - k_z z)}
\]
(A.71)
to (A.70) and then ordering,
\[
\frac{r}{R} \frac{dR}{dr} + \frac{r^2}{R} \frac{d^2 R}{dr^2} + \frac{r^2}{R} (k^2 - k_z^2) = -\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2}
\]
(A.72)
is obtained. Since left-hand member is a function of \(r\) only, while the right hand member is a function of \(\phi\) only, this value must be a constant. Representing the constant as \(m^2\), it is written as
\[
\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = -m^2
\]
\[
\Phi = C \cos(m\phi).
\]
(A.73)
Applying (A.73) to (A.70),
\[ r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + \left[ r^2 (k^2 - k_z^2) - m^2 \right] R = 0 \]  
(A.74)
is obtained. This is a Bessel differential equation. By choosing a solution which is finite at \( r = 0 \),
the solution is a Bessel Function and would be represented as
\[ R(r) = A J_m(r \sqrt{k^2 - k_z^2}). \]  
(A.75)
Through these discussions, \( E_z \) is represented as
\[ E_z = E_0 J_m(r \sqrt{k^2 - k_z^2}) \cos(m\phi)e^{i(\omega t + k_z z)}. \]  
(A.76)
Since \( E_z \) is 0 at \( r = a \) from boundary condition,
\[ J_m(a \sqrt{k^2 - k_z^2}) = 0 \]
\[ \implies \sqrt{k^2 - k_z^2} = \frac{j_m}{a} \]  
(A.77)
is obtained, where \( j_m \) is the \( n \)th root of Bessel Function \( J_m \). This leads to
\[ E_z = E_0 J_m(r \sqrt{k^2 - k_z^2}) \cos(m\phi)e^{i(\omega t + k_z z)} \]  
(A.78)
\[ E_r = \pm i \frac{k_z}{\sqrt{k^2 - k_z^2}} E_0 J'_m(r \sqrt{k^2 - k_z^2}) \cos(m\phi)e^{i(\omega t + k_z z)} \]  
(A.79)
\[ E_\phi = \pm i \frac{k_z m}{k^2 - k_z^2} E_0 J_m(r \sqrt{k^2 - k_z^2}) \sin(m\phi)e^{i(\omega t + k_z z)}, \]  
(A.80)
where \( \prime \) represents derivation in \( r \). The actual field would be the superposition of electromagnetic
wave transmitting to plus and minus direction in \( z \), and therefore they are written as
\[ E_z = E_0 J_m(r \sqrt{k^2 - k_z^2}) \cos(m\phi)(A_-e^{-ik_z z} + A_+e^{ik_z z})e^{i\omega t} \]  
(A.81)
\[ E_r = i \frac{k_z}{\sqrt{k^2 - k_z^2}} E_0 J'_m(r \sqrt{k^2 - k_z^2}) \cos(m\phi)(-A_-e^{-ik_z z} + A_+e^{ik_z z})e^{i\omega t} \]  
(A.82)
\[ E_\phi = i \frac{k_z m}{k^2 - k_z^2} E_0 J_m(r \sqrt{k^2 - k_z^2}) \sin(m\phi)(-A_-e^{-ik_z z} + A_+e^{ik_z z})e^{i\omega t}. \]  
(A.83)
Since boundary condition is given as
\[ E_r(z = 0) = 0 \implies A_- = A_+ \]  
(A.84)
\[ E_r(z = L) = 0 \implies k_z = \frac{l\pi}{L}, \]  
(A.85)
TM\(_{mn}\) mode of a cylindrical cavity is finally represented as
\[ E_z = E_0 J_m(r \sqrt{k^2 - k_z^2}) \cos(m\phi)e^{i\omega t} \]  
(A.86)
\[ E_r = -i \frac{k_z}{\sqrt{k^2 - k_z^2}} E_0 J'_m(r \sqrt{k^2 - k_z^2}) \cos(m\phi)\sin \left( \frac{l\pi}{L} \right) e^{i\omega t} \]  
(A.87)
\[ E_\phi = -i \frac{k_z m}{k^2 - k_z^2} E_0 J_m(r \sqrt{k^2 - k_z^2}) \sin(m\phi)\sin \left( \frac{l\pi}{L} \right) e^{i\omega t} \]  
(A.88)
\[ H_z = 0 \]  
(A.89)
\[ H_r = -i \frac{\omega e_0}{k^2 - k_z^2} m E_0 J_m(r \sqrt{k^2 - k_z^2}) \sin(m\phi)\cos \left( \frac{l\pi}{L} \right) e^{i\omega t} \]  
(A.90)
\[ H_\phi = -i \frac{\omega e_0}{\sqrt{k^2 - k_z^2}} E_0 J'_m(r \sqrt{k^2 - k_z^2}) \cos(m\phi)\cos \left( \frac{l\pi}{L} \right) e^{i\omega t}. \]  
(A.91)
It is easily shown that $E_{zz}$ and $H_{\perp}$ are zero at cavity surface, which are the requirements from the boundary condition.

Also, by using $\omega = 2\pi f = ck$, (A.77), and (A.85) the resonant frequency is represented as

$$f = \frac{1}{2\pi} c \sqrt{(k^2 - k_z^2) + k_z^2} = \frac{c}{2\pi} \sqrt{\left(\frac{jmn}{R}\right)^2 + \left(\frac{l\pi}{L}\right)^2}.$$  \hfill (A.92)

### A.2 Calculation of Energy Loss at Cavity Wall

#### A.2.1 Starting point — Maxwell Equation

Maxwell equation of a conductor is represented as

$$\nabla \times \mathbf{E} + \mu_0 \frac{\partial \mathbf{H}}{\partial t} = 0 \quad \text{(A.93)}$$

$$\nabla \cdot \mathbf{H} = 0 \quad \text{(A.94)}$$

$$\nabla \times \mathbf{H} - \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} = \sigma \mathbf{E} \quad \text{(A.95)}$$

$$\nabla \cdot \mathbf{E} = 0. \quad \text{(A.96)}$$

(A.96) works out since there is no true charge in a conductor. One must be aware that $\mathbf{E}$ and $\mathbf{H}$ are the field of the conductor and not the RF, which is represented in (A.1) \text{ to } (A.4).

Since $\mathbf{J}$ is parallel to the conductor surface, $\mathbf{E}$ is also parallel. From the boundary condition, only $\mathbf{E}_{\perp}$ and $\mathbf{H}_{\perp}$ exist on a conductor surface, and $\mathbf{E}_{zz}$ is zero. This is because considering a perfect conductor $\sigma$ is $\infty$, and in order to suppress $\mathbf{J}$ to a finite value $\mathbf{E} = 0$ is required. However in practical, metals have a finite $\sigma$ and therefore $\mathbf{E}$ exists.

By calculating $\nabla \times$ (A.93) and applying (A.96),

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$$

$$\nabla \times (-\mu_0 \frac{\partial \mathbf{H}}{\partial t}) = -\mu \varepsilon \frac{\partial \mathbf{E}}{\partial t} - \varepsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$\leftrightarrow \nabla^2 \mathbf{E} = \mu \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \varepsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \text{(A.97)}$

is obtained. Good conductors such as copper have large $\sigma$, and thus $\mu \varepsilon$ is negligible. Therefore, (A.97) is represented as

$$\nabla^2 \mathbf{E} = \mu \varepsilon \frac{\partial \mathbf{E}}{\partial t}. \quad \text{(A.98)}$$

Let’s take the direction of surface current $\mathbf{J}$ as $z$ direction, the other parallel direction as $x$ direction and the transverse direction as $y$ direction. In this situation, $\mathbf{E}$ is represented as (0, 0, $E_z$), and $\mathbf{H}$ as ($H_x$, 0, 0). $E_z$ is represented as

$$E_z = E_0 e^{i(\omega t - k_x x)}. \quad \text{(A.99)}$$

Considering (A.96),

$$\frac{\partial E_z}{\partial z} = 0 \quad \text{(A.100)}$$

is obtained. Therefore, $E_z$ does not depend on $z$, and $\mathbf{k}$ is represented as ($k_x$, $k_y$, 0). Applying these expressions to (A.98),

$$k_x^2 + k_y^2 = -i\omega \sigma \mu \quad \text{(A.101)}$$
is obtained. Wave number $k$ is an inverse of the wave length $\lambda$. Since $\lambda_x$ and $\lambda_y$ are the wave length in free space and inside the conductor respectively, $\lambda_x >> \lambda_y$ works out. Therefore $k_x$ is negligible compared with $k_y$. This leads to,

$$k_y = \pm \sqrt{-i\omega\sigma\mu} = \pm (i-1) \sqrt{\frac{\omega\sigma\mu}{2}}.$$  
(A.102)

By choosing a decaying solution, (A.99) is written as

$$E_z = E_0 e^{i(\omega t - \sqrt{\frac{\sigma}{2\mu}} y)} e^{-\sqrt{\frac{\sigma}{2\mu}} y}.$$  
(A.103)

It is easily shown that when depth $y$ in the conductor becomes

$$y = \delta \equiv \sqrt{\frac{2}{\omega \sigma \mu}},$$  
(A.104)

field $E_z$ decays to $1/e$. $\delta$ is called the skin depth.

Assuming that displacement current is negligible, (A.99) is represented as

$$\nabla \times \mathbf{H} = \sigma \mathbf{E}.$$  
(A.105)

Applying $\mathbf{H} = (H_x, 0, 0)$ and $\mathbf{E} = (0, 0, E_z)$ to (A.105),

$$-\frac{\partial H_x}{\partial y} = \sigma E_z$$

$$= \sigma E_0 e^{i(\omega t - \frac{y}{\sigma})} e^{-\frac{y}{\sigma}}$$

$$\leftarrow H_x = -\frac{\delta}{1 + i} \sigma E_0 e^{i(\omega t - \frac{y}{\sigma})} e^{-\frac{y}{\sigma}}$$  
(A.106)

is obtained. Therefore, by defining the amplitude of $H_x$ at conductor surface as

$$H_0 \equiv \frac{-\delta}{1 + i} \sigma E_0 e^{-\frac{y}{\sigma}},$$  
(A.107)

$H_x$ and $E_z$ are represented as

$$H_x = H_0 e^{i(\omega t - \frac{y}{\sigma})}.$$  
(A.108)

$$E_z = -\frac{1 + i}{\sigma \delta} H_0 e^{i(\omega t - \frac{y}{\sigma})}.$$  
(A.109)

Average energy flow density of electromagnetic field per unit time is calculated from

$$\bar{\mathbf{S}} = \text{Re} \left[ \frac{1}{2} \mathbf{E} \times \mathbf{H}^* \right],$$  
(A.110)

where $\mathbf{S}$ is the poynting vector. Applying (A.108) and (A.109) to (A.110),

$$\bar{\mathbf{S}} = E_z H_x = -\frac{1}{2\sigma \delta} |H_0|^2 = -\frac{1}{2\sigma \delta} |H_x|^2$$  
(A.111)

is obtained. The sign "-" represents the energy loss. Therefore, energy loss at the cavity surface $P$ is proved to be

$$P = \frac{1}{2\sigma \delta} |H_x|^2 = \frac{1}{2} R_s |H_z|^2.$$  
(A.112)

Please notice that energy flow from $E_\perp \times H_\parallel$ is parallel to the surface and is not lost. Only the considering $E_\parallel \times H_\parallel$ is transverse to the cavity surface, which is lost through the inner wall.
A.3 HFSS Field Calculator

A.3.1 Stored Energy $U$

1. Select "$E < E_x, E_y, E_z >" from "Quantity".
2. Select "0" from "Number".
3. Select "At Phase" from "Complex" and phase of the field $E < E_x, E_y, E_z >$ is set to zero.
4. Apply "Mag" to acquire $|E|$.
5. Select "2" from "Number".
6. Apply "Pow" and acquire $|E|^2$.
7. Select "$\varepsilon_0" from "Constant".
8. Apply "$\mu_0$" to calculate $\mu_0 |H|^2$.
9. Calculate $\mu_0 |H|^2$ as well.
10. Apply "$+" to calculate $(\varepsilon_0 |E|^2 + \mu_0 |H|^2)$.
11. Select the whole cavity from "Volume" at "Geometry".
12. Apply "Integrate" to assign the cavity as the volume of integration.
13. Perform "Eval" and calculate $\int_V (\varepsilon_0 |E|^2 + \mu_0 |H|^2) dV$.

A.3.2 Field Integration along axis

1. Select "$E < E_x, E_y, E_z >" from "Quantity".
2. Select "0" from "Number".
3. Select "At Phase" from "Complex" and phase of the field $E < E_x, E_y, E_z >$ is set to zero.
4. Select the considering polyline from "Line" at "Geometry".
5. Apply "Tangent" and acquire tangential component of the field.
6. Apply "Integrate" to integrate the tangent along the selected line.
7. Perform "Eval" and calculate $\int E_z dz$.

A.4 Slater’s Perturbation Theory

A.4.1 Vector Formulas

Please note that these vector formulas are used in the following discussion without noticing.

\[
\nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A \tag{A.113}
\]
\[
\nabla \times (\nabla f) = 0 \tag{A.114}
\]
\[
A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B) \tag{A.115}
\]
\[
\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B) \tag{A.116}
\]
\[
\int_V \nabla \cdot A dV = \int_S A \cdot n dS \quad (\text{Gauss's Law}) \tag{A.117}
\]
Appendix 119

A.4.2 Orthogonal function expansion of electro-magnetic field of a resonant cavity

According to Helmholtz Theorem, every vector quantity \( \mathbf{A} \) can be represented as

\[
\mathbf{A} = \mathbf{A}^0 + \mathbf{A}'
\]

\[
\nabla \cdot \mathbf{A}^0 = 0, \quad \nabla \times \mathbf{A}' = 0.
\]

(A.118)

Therefore, electro-magnetic field of a cavity is represented as

\[
\mathbf{E} = \mathbf{E}^0 + \mathbf{E}', \quad \nabla \cdot \mathbf{E}^0 = 0, \quad \nabla \times \mathbf{E}' = 0
\]

\[
\mathbf{H} = \mathbf{H}^0 + \mathbf{H}', \quad \nabla \cdot \mathbf{H}^0 = 0, \quad \nabla \times \mathbf{H}' = 0.
\]

(A.119)

Applying these to Maxwell Equation (A.1) and calculating rotation,

\[
\nabla \times (\nabla \times \mathbf{E}) + \mu \frac{\partial}{\partial t} (\nabla \times \mathbf{H}) = 0
\]

(A.120)
is obtained. The left hand term is

\[
\nabla \times (\nabla \times \mathbf{E}) = \nabla \times (\nabla \times (\mathbf{E}^0 + \mathbf{E}')) = \nabla \times (\nabla \times \mathbf{E}^0)
\]

\[
= \nabla (\nabla \cdot \mathbf{E}^0) - \nabla^2 \mathbf{E}^0 = -\nabla^2 \mathbf{E}^0,
\]

(A.121)

while the right hand term is

\[
\mu \frac{\partial}{\partial t} (\nabla \times \mathbf{H}) = \mu \frac{\partial}{\partial t} (\nabla \times (\mathbf{H}^0 + \mathbf{H}'))
\]

\[
= \mu \frac{\partial}{\partial t} (\nabla \times \mathbf{H}^0) = \nabla \times \mu \frac{\partial}{\partial t} \mathbf{H}^0.
\]

(A.122)

Assuming that \( \mathbf{E}^0, \mathbf{H}^0 \) are represented as

\[
\mathbf{E}^0 \propto e^{i(\omega t - kx)}
\]

(A.123)

\[
\mathbf{H}^0 \propto e^{i(\omega t - kx)},
\]

(A.124)

where \( \omega = ck, \ c = \frac{1}{\sqrt{\varepsilon \mu}} \) works out, (A.120) is calculated as

\[
- \varepsilon \omega \mathbf{E}^0 = i(\nabla \times \mathbf{H}^0).
\]

(A.125)

In the same way,

\[
\mu \omega \mathbf{H}^0 = i(\nabla \times \mathbf{E}^0).
\]

(A.126)
is also obtained. As one can see, \( \mathbf{E}^0 \) and \( \mathbf{H}^0 \) are related through (A.125) and (A.126).

First, let’s consider \( \mathbf{E}^0 \) and \( \mathbf{H}^0 \). By introducing electro-magnetic field of mode \( a \)

\[
\mathbf{E}_a \equiv \sqrt{\varepsilon} \mathbf{E}^0
\]

(A.127)

\[
\mathbf{H}_a \equiv -i \sqrt{\mu} \mathbf{H}^0,
\]

(A.128)

(A.125) and (A.126) are represented as

\[
k_a \mathbf{E}_a = \nabla \times \mathbf{H}_a
\]

(A.129)

\[
k_a \mathbf{H}_a = \nabla \times \mathbf{E}_a,
\]

(A.130)
where $\omega_a = ck_a$. Moreover,

$$\nabla \times (\nabla \times E_a) = \nabla (\nabla \cdot E_a) - \nabla^2 E_a = -\nabla^2 E_a$$

$$\leftrightarrow \nabla \times k_a H_a = -\nabla^2 E_a$$

$$\leftrightarrow \nabla^2 E_a + k_a^2 E_a^2 = 0$$  \hspace{1cm} (A.131)

is obtained, and also by the same method

$$\nabla^2 H_a + k_a^2 H_a^2 = 0$$  \hspace{1cm} (A.132)

is obtained.

Next, $E'$ is considered. Since $E'$ satisfies $\nabla \times E' = 0$, it is written as

$$E' = -\nabla \varphi.$$  \hspace{1cm} (A.133)

By introducing electronic filed and scalar potential of mode $a$

$$F_a = \sqrt{\gamma} E'$$ \hspace{1cm} (A.134)

$$\phi_a = -k_a \sqrt{\gamma} \varphi.$$ \hspace{1cm} (A.135)

(A.133) is represented as

$$k_a F_a = \nabla \phi_a.$$ \hspace{1cm} (A.136)

Finally, $H'$, which is the most simple term, is considered. Since Maxwell Equation requires \hspace{1cm} (A.2), this term must be equivalent to zero.

Two boundary conditions can be considered for $E_a$, $F_a$ and $H_a$. The first one is called the "short boundary condition", represented by surface $S$. The boundary condition on $S$ is

$$n \times E_a = 0, \quad n \cdot H_a = 0, \quad n \times F_a = 0, \quad \phi_a = 0.$$ \hspace{1cm} (A.137)

As shown in (A.137), electric field parallel to the boundary surface $E_{\|}$ and magnetic field transverse to the surface $H_{\perp}$ are zero. This boundary condition corresponds to that of the perfect conductor. On the other hand what called "open boundary condition", represented by surface $S'$, can also be defined. The boundary condition on $S'$ is

$$n \cdot E_a = 0, \quad n \times H_a = 0, \quad n \times F_a = 0, \quad \phi_a = 0.$$ \hspace{1cm} (A.138)

Therefore, a cavity volume $V$ surrounded by $S$ and $S'$ is going to be considered.

Normalized orthogonality of these functions in a cavity must be proved next. Those are,

$$\int_V E_a \cdot E_b dV = \delta_{a,b}$$ \hspace{1cm} (A.139)

$$\int_V H_a \cdot H_b dV = \delta_{a,b}$$ \hspace{1cm} (A.140)

$$\int_V F_a \cdot F_b dV = \delta_{a,b}$$ \hspace{1cm} (A.141)

$$\int_V F_a \cdot E_b dV = 0.$$ \hspace{1cm} (A.142)

At first, (A.139) is considered. Let’s consider the following equation,

$$k_a \nabla \cdot (E_b \times H_a) - k_b \nabla \cdot (E_a \times H_b) = \nabla \cdot [E_b \times (\nabla \times E_a)] - \nabla \cdot [E_a \times (\nabla \times E_b)]$$

$$= (\nabla \times E_a) \cdot (\nabla \times E_b) - E_b \cdot [\nabla \times (\nabla \times E_a)]$$

$$- (\nabla \times E_b) \cdot (\nabla \times E_a) - E_a \cdot [\nabla \times (\nabla \times E_b)]$$

$$= E_a \cdot [\nabla \times (\nabla \times E_b)] - E_b \cdot [\nabla \times (\nabla \times E_a)]$$

$$= E_a \cdot (-\nabla^2 E_b) - E_b \cdot (-\nabla^2 E_a)$$

$$= (k_b^2 - k_a^2) E_a E_b.$$ \hspace{1cm} (A.143)
By integrating (A.143) in the cavity volume,

\[ (k_a^2 - k_a^2) \int_V \mathbf{E}_a \mathbf{E}_b dV = \int_V [k_a \nabla \cdot (\mathbf{E}_b \times \mathbf{H}_a) - k_b \nabla \cdot (\mathbf{E}_a \times \mathbf{H}_b)] \]

\[ = \int_{S,S'} \mathbf{n} \cdot [k_a \nabla \cdot (\mathbf{E}_b \times \mathbf{H}_a) - k_b \nabla \cdot (\mathbf{E}_a \times \mathbf{H}_b)] dS \]

\[ = \int_S [k_a \mathbf{H}_a \cdot (\mathbf{n} \times \mathbf{E}_b) - k_b \mathbf{H}_b \cdot (\mathbf{n} \times \mathbf{E}_a)] dS \]

\[ + \int_{S'} [k_b \mathbf{E}_a \cdot (\mathbf{n} \times \mathbf{H}_b) - k_a \mathbf{E}_b \cdot (\mathbf{n} \times \mathbf{H}_a)] dS' \]

\[ = 0 \quad (a \neq b) \]  

(A.144)

is obtained. Please note that boundary condition (A.137) and (A.138) are applied. This is the same case with \( \mathbf{H}_a \). Therefore by normalizing as

\[ \int_V \mathbf{E}_a^2 dV = 1 \]  

(A.145)

\[ \int_V \mathbf{H}_a^2 dV = 1, \]  

(A.146)

(A.139) and (A.140) are proved. However, since \( \mathbf{E}_a \) and \( \mathbf{H}_a \) are related through (A.125) and (A.126), we must check if they can actually be normalized at the same time. For this purpose,

\[ \nabla \cdot (\mathbf{E}_a \times (\nabla \times \mathbf{E}_a)) = (\nabla \times \mathbf{E}_a) \cdot (\nabla \times \mathbf{E}_a) - \mathbf{E}_a \cdot \nabla \times (\nabla \times \mathbf{E}_a) \]

\[ = (k_a \mathbf{H}_a)^2 - \mathbf{E}_a \cdot (\nabla \times k_a \mathbf{H}_a) \]

\[ = k_a^2 (\mathbf{H}_a^2 - \mathbf{E}_a^2) \]  

(A.147)

is integrated over cavity volume. Thus,

\[ k_a^2 \left( \int_V \mathbf{H}_a^2 dV - \int_V \mathbf{E}_a^2 dV \right) = \int_V \nabla \cdot (\mathbf{E}_a \times (\nabla \times \mathbf{E}_a)) dV \]

\[ = \int_{S,S'} \mathbf{n} \cdot k_a (\mathbf{E}_a \times \mathbf{H}_a) dS \]

\[ = \int_S k_a \mathbf{H}_a \cdot (\mathbf{n} \times \mathbf{E}_a) dS - \int_{S'} k_a \mathbf{E}_a \cdot (\mathbf{n} \times \mathbf{H}_a) dS' \]

\[ = 0 \]  

(A.148)

is obtained, which shows that \( \mathbf{E}_a \) and \( \mathbf{H}_a \) can be normalized at the same time.

Next, (A.141) is considered. Assuming \( \phi_a (x, t) = e^{i \omega t} \phi_a (x) \), Helmholtz Equation

\[ \nabla^2 \phi_a + k_a^2 \phi_a = 0 \]  

(A.149)

is satisfied. Therefore,

\[ \nabla \cdot (\phi_a \nabla \phi_b) = \phi_a \nabla^2 \phi_b + \nabla \phi_a \cdot \nabla \phi_b \]

\[ = \phi_a (-k_a^2 \phi_a) + \nabla \phi_a \cdot \nabla \phi_b \]  

(A.150)

\[ \nabla \cdot (\phi_b \nabla \phi_a) = \phi_b (-k_a^2 \phi_a) + \nabla \phi_b \cdot \nabla \phi_a \]  

(A.151)

are obtained. By integrating (A.150) – (A.151) over \( V \),

\[ (k_a^2 - k_b^2) \int_V \phi_a \phi_b dV = \int_V [\nabla \cdot (\phi_a \nabla \phi_b) - \nabla \cdot (\phi_b \nabla \phi_a)] dV \]

\[ = \int_{S,S'} [\phi_a \nabla \phi_b \cdot \mathbf{n} - (\phi_b \nabla \phi_a) \cdot \mathbf{n}] dS \]

\[ = 0 \quad (a \neq b) \]  

(A.152)
is obtained. Please note that boundary condition (A.137) and (A.138) are applied. Also, by applying
\[ \int_V \nabla \cdot (\phi_a \nabla \phi_b) dV = \int_{S, S'} (\phi_a \nabla \phi_b) \cdot n dS = 0 \] (A.153)
and (A.152) to volume integral of (A.150),
\[ \int_V \nabla \phi_a \cdot \nabla \phi_b dV = \int_V k_a F_a \cdot k_b F_b dV = 0 \quad (a \neq b) \] (A.154)
is obtained, and thus (A.141) is proved. Also by applying \( a = b \) to volume integral of (A.150),
\[-k_a^2 \int_V \phi_a^2 dV + \int_V \nabla \phi_a \cdot \nabla \phi_a dV = 0 \]
\[ \quad \Longleftrightarrow -\int_V \phi_a^2 dV + \int_V F_a^2 dV = 0 \] (A.155)
is obtained, which shows that \( F_a \) and \( \phi_a \) can be normalized at the same time.
Finally, (A.142) is considered. By integrating
\[ \nabla \cdot (\phi_a E_b) = \phi_a \nabla \cdot E_b + (\nabla \phi_a) \cdot E_b = k_a F_a \cdot E_b \] (A.156)
over \( V \),
\[ k_a \int_V F_a \cdot E_b dV = \int_V \nabla \cdot (\phi_a E_b) dV = \int_{S, S'} (\phi_a E_b) \cdot n dS = 0 \] (A.157)
is obtained, and therefore (A.142) is proved.
Thus, arbitrary electro-magnetic field in the cavity is represented as
\[ E = \sum_a e_a E_a + \sum_a f_a F_a \] (A.158)
\[ H = \sum_a h_a H_a \] (A.159)
where \( a \) represents the resonance mode of the cavity. Applying (A.139) \( \sim \) (A.142),
\[ e_a = \int_V E \cdot E_a dV \] (A.160)
\[ f_a = \int_V F \cdot F_a dV \] (A.161)
\[ h_a = \int_V H \cdot H_a dV \] (A.162)
are also obtained.

### A.4.3 Maxwell Equation in a resonant cavity

Electric field \( E \), magnetic field \( H \), current \( J \) and charge density \( \rho \) of a resonant cavity of volume \( V \) can be expanded with \( E_a, F_a, H_a \) and \( \phi_a \), as
\[ E = \sum_a \left( E_a \int_V E \cdot E_a dV + F_a \int_V E \cdot F_a dV \right) \] (A.163)
\[ H = \sum_a H_a \int_V H \cdot H_a dV \] (A.164)
\[ J = \sum_a \left( E_a \int_V J \cdot E_a dV + F_a \int_V J \cdot F_a dV \right) \] (A.165)
\[ \rho = \sum_a \int_V \rho \phi_a dV \] (A.166)
respectively. Using these expressions, let’s derivate Maxwell Equations for resonant cavities. At first, to consider \( (A.1) \), \( \nabla \times \mathbf{E} \) is expanded as

\[
\nabla \times \mathbf{E} = \Sigma_a \mathbf{H}_a \int_V (\nabla \times \mathbf{E}) \cdot \mathbf{H}_a dV. \tag{A.167}
\]

To determine the coefficient of the expansion,

\[
\nabla \cdot [\mathbf{E} \times (\nabla \times \mathbf{E}_a)] = (\nabla \times \mathbf{E}_a) \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot [\nabla \times (\nabla \times \mathbf{E}_a)] = k_a \mathbf{H}_a \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot [\nabla \times k_a \mathbf{H}_a] = k_a \mathbf{H}_a \cdot (\nabla \times \mathbf{E}) - k_a^2 \mathbf{E} \mathbf{E}_a \tag{A.168}
\]

is integrated over cavity volume \( V \), which leads to

\[
k_a \int_V \mathbf{H}_a \cdot (\nabla \times \mathbf{E}) dV - k_a^2 \int_V \mathbf{E} \mathbf{E}_a dV = \int_V (\nabla \cdot [\mathbf{E} \times (\nabla \times \mathbf{E}_a)]) dV = \int_S [\mathbf{E} \times k_a \mathbf{H}_a] \cdot \mathbf{n} dS + \int_{S'} [\mathbf{E} \times k_a \mathbf{H}_a] \cdot \mathbf{n} dS' = k_a \int_S (\mathbf{n} \times \mathbf{E}) \cdot \mathbf{H}_a dS + k_a \int_{S'} (\mathbf{n} \times \mathbf{E}) \cdot \mathbf{H}_a dS'. \tag{A.169}
\]

Considering boundary conditions,\[
\int_V \mathbf{H}_a \cdot (\nabla \times \mathbf{E}) dV = k_a \int_V \mathbf{E} \cdot \mathbf{E}_a dV + \int_S (\mathbf{n} \times \mathbf{E}) \cdot \mathbf{H}_a dS \tag{A.170}
\]

is obtained. Therefore, (A.167) is written as

\[
\nabla \times \mathbf{E} = \Sigma_a \mathbf{H}_a \left( k_a \int_V \mathbf{E} \cdot \mathbf{E}_a dV + \int_S (\mathbf{n} \times \mathbf{E}) \cdot \mathbf{H}_a dS \right). \tag{A.171}
\]

Also,

\[
\nabla \times \mathbf{H} = \Sigma_a \mathbf{E}_a \left( k_a \int_V \mathbf{H} \cdot \mathbf{H}_a dV + \int_{S'} (\mathbf{n} \times \mathbf{H}) \cdot \mathbf{E}_a dS \right) \tag{A.172}
\]

is obtained from similar calculation.

By applying expression (A.171) and (A.164) to (A.1),

\[
\Sigma_a \mathbf{H}_a \left( k_a \int_V \mathbf{E} \cdot \mathbf{E}_a dV + \int_S (\mathbf{n} \times \mathbf{E}) \cdot \mathbf{H}_a dS \right) + \mu \frac{\partial}{\partial t} \Sigma_a \mathbf{H}_a \int_V \mathbf{H} \cdot \mathbf{H}_a dV = 0 \tag{A.173}
\]

is obtained. By comparing coefficients of \( \mathbf{H}_a \),

\[
k_a \int_V \mathbf{E} \cdot \mathbf{E}_a dV + \mu \frac{\partial}{\partial t} \int_V \mathbf{H} \cdot \mathbf{H}_a dV = - \int_S (\mathbf{n} \times \mathbf{E}) \cdot \mathbf{H}_a dS \tag{A.174}
\]

is obtained. Also by applying expression (A.172) and (A.164) to (A.3) and comparing the coefficients of \( \mathbf{E}_a \) and \( \mathbf{F}_a \),

\[
k_a \int_V \mathbf{H} \cdot \mathbf{H}_a dV - \epsilon \frac{\partial}{\partial t} \int_V \mathbf{E} \cdot \mathbf{E}_a dV = \int_V \mathbf{J} \cdot \mathbf{E}_a dV - \int_{S'} (\mathbf{n} \times \mathbf{H}) \cdot \mathbf{E}_a dS \tag{A.175}
\]

\[
-\epsilon \frac{\partial}{\partial t} \int_V \mathbf{E} \cdot \mathbf{F}_a dV = \int_V \mathbf{J} \cdot \mathbf{F}_a dV \tag{A.176}
\]
are obtained.
Next, \((A.4)\) is considered. In order to obtain the series expansion of \(\nabla \cdot \mathbf{E}\), the following formula
\[
\nabla \cdot (\phi_a \mathbf{E}) = \phi_a \nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla \phi_a = \phi_a \nabla \cdot \mathbf{E} + \mathbf{E} \cdot \mathbf{F}_a
\]  
(A.177)
is considered. By integrating \((A.177)\) over \(V\),
\[
\int_V \phi_a \nabla \cdot \mathbf{E} dV + k_a \int_V \mathbf{E} \cdot \mathbf{F}_a dV = \int_V \nabla \cdot (\phi_a \mathbf{E}) dV = \int_{S,S'} \mathbf{n} \cdot (\phi_a \mathbf{E}) dS
\]  
(A.178)
is obtained. By applying boundary conditions, the right hand term is zero and therefore
\[
\int_V (\nabla \cdot \mathbf{E}) \phi_a dV = -k_a \int_V \mathbf{E} \cdot \mathbf{F}_a dV
\]  
(A.179)
is obtained. As a result, \(\nabla \cdot \mathbf{E}\) is expanded as
\[
\nabla \cdot \mathbf{E} = \Sigma_a \phi_a \left[ -k_a \int_V \mathbf{E} \cdot \mathbf{F}_a dV \right].
\]  
(A.180)
Applying \((A.180)\) and \((A.166)\) to \((A.4)\),
\[
\Sigma_a \phi_a \left[ -k_a \int_V \mathbf{E} \cdot \mathbf{F}_a dV \right] = \frac{1}{\epsilon} \Sigma_a \int_V \rho \phi_a dV
\]  
(A.181)
is obtained. By comparing the coefficients,
\[
-k_a \epsilon \int_V \mathbf{E} \cdot \mathbf{F}_a dV = \int_V \rho \phi_a dV
\]  
(A.182)
is obtained. At this point, by calculating \(k_a \times (A.176) - \frac{\partial}{\partial t} (A.182)\),
\[
k_a \int_V \mathbf{J} \cdot \mathbf{F}_a dV = \frac{\partial}{\partial t} \int_V \rho \phi_a dV
\]  
(A.183)
is obtained. However, this is an identical equation. By integrating
\[
\nabla \cdot (\phi_a \mathbf{J}) = \phi_a \nabla \cdot \mathbf{J} + \mathbf{J} \cdot \nabla \phi_a = \phi_a \nabla \cdot \mathbf{J} + k_a \mathbf{J} \cdot \mathbf{F}_a
\]  
(A.184)
over volume \(V\),
\[
\int_V \phi_a \nabla \cdot \mathbf{J} dV + k_a \int_V \mathbf{J} \cdot \mathbf{F}_a dV = \int_V \nabla \cdot (\phi_a \mathbf{J}) dV = \int_{S,S'} \mathbf{n} \cdot (\phi_a \mathbf{J}) dS = 0
\]  
(A.185)
is obtained. Therefore, \((A.183)\) is written as
\[
\int_V (\nabla \cdot \mathbf{J}) \phi_a dV = \frac{\partial}{\partial t} \int_V \rho \phi_a dV,
\]  
(A.186)
which represents the current continuity equation. Thus, \((A.176)\) is enough to describe \((A.183)\).
From \((A.174)\) and \((A.175)\), wave equations of electro-magnetic field are represented as
\[
\epsilon \frac{\partial^2}{\partial t^2} \int_V \mathbf{E} \cdot \mathbf{E}_a dV + k_a^2 \int_V \mathbf{E} \cdot \mathbf{E}_a dV = -\mu \frac{\partial}{\partial t} \left[ \int_V \mathbf{J} \cdot \mathbf{E}_a dV - \int_{S'} (\mathbf{n} \times \mathbf{H}) \cdot \mathbf{E}_a dS' \right] - k_a \int_S (\mathbf{n} \times \mathbf{E}) \cdot \mathbf{H}_a dS
\]  
(A.187)
\[
\epsilon \frac{\partial^2}{\partial t^2} \int_V \mathbf{H} \cdot \mathbf{H}_a dV + k_a^2 \int_V \mathbf{H} \cdot \mathbf{H}_a dV = -\epsilon \frac{\partial}{\partial t} \int_{S'} (\mathbf{n} \times \mathbf{E}) \cdot \mathbf{H}_a dS + k_a \int_V \mathbf{J} \cdot \mathbf{E}_a dV - k_a \int_{S'} (\mathbf{n} \times \mathbf{H}) \cdot \mathbf{E}_a dS'.
\]  
(A.188)
A.4.4 Bead Perturbation in a Resonant Cavity

Considering a cavity consisting of a perfect conductor, the boundary condition would be \( S \) at all boundaries. Assume that mode \( a \) is dominant in the cavity, and \( \mathbf{J} = 0 \). In this case, the initial electro-magnetic field is represented as

\[
\mathbf{E}_0 = \mathbf{E}_a \int_V \mathbf{E}_0 \cdot \mathbf{E}_a dV
\]

\[
\mathbf{H}_0 = \mathbf{H}_a \int_V \mathbf{H}_0 \cdot \mathbf{H}_a dV,
\]

and thus (A.187) and (A.188) are written as

\[
\varepsilon \mu \frac{\partial^2}{\partial t^2} \int_V \mathbf{E}_0 \cdot \mathbf{E}_a dV + k_a^2 \int_V \mathbf{E}_0 \cdot \mathbf{E}_a dV = 0
\]

(A.191)

\[
\varepsilon \mu \frac{\partial^2}{\partial t^2} \int_V \mathbf{H}_0 \cdot \mathbf{H}_a dV + k_a^2 \int_V \mathbf{H}_0 \cdot \mathbf{H}_a dV = 0.
\]

(A.192)

Please note that boundary condition (A.137) is applied.

Now, let’s consider a small perturbation, for example a sphere bead, is input to the cavity. The field inside the cavity after introducing the perturbation is written as

\[
\mathbf{E} = \mathbf{E}_a \int_V \mathbf{E} \cdot \mathbf{E}_a dV
\]

(A.193)

\[
\mathbf{H} = \mathbf{H}_a \int_V \mathbf{H} \cdot \mathbf{H}_a dV,
\]

(A.194)

assuming that mode \( a \) is still dominant.

The bead surface and volume are defined as \( S_p \) and \( V_p \), respectively. Since electro-magnetic field is considered to be discontinuous at \( S_p \), (A.137) does not work on \( S_p \). Therefore boundary \( S_p \) must be considered additionally to (A.191) and (A.192). Referring (A.188), their contribution would be represented as

\[
\varepsilon \mu \frac{\partial^2}{\partial t^2} \int_V \mathbf{H} \cdot \mathbf{H}_a dV + k_a^2 \int_V \mathbf{H} \cdot \mathbf{H}_a dV = -k_a \int_{S_p} (\mathbf{n} \times \mathbf{H}) \cdot \mathbf{E}_a dS_p.
\]

(A.195)

The perturbation term can be calculated as

\[
\int_{S_p} (\mathbf{n} \times \mathbf{H}) \cdot \mathbf{E}_a dS_p = \int_{S_p} (\mathbf{n} \times \mathbf{H}_a) \cdot \mathbf{E}_a dS_p \cdot \int_V \mathbf{H} \cdot \mathbf{H}_a dV
\]

\[
= \int_{S_p} [-\mathbf{n} \cdot (\mathbf{E}_a \times \mathbf{H}_a)] dS_p \cdot \int_V \mathbf{H} \cdot \mathbf{H}_a dV.
\]

(A.196)

Please note that \( \mathbf{n} \) represent the unit normal vector of the cavity. As shown in Figure A.1, the unit normal vector of the bead is thus represented as \( -\mathbf{n} \). Therefore,

\[
\int_{S_p} (\mathbf{n} \times \mathbf{H}) \cdot \mathbf{E}_a dS_p = \int_{V_p} \nabla \cdot (\mathbf{E}_a \times \mathbf{H}_a) dV_p \cdot \int_V \mathbf{H} \cdot \mathbf{H}_a dV
\]

\[
= \int_{V_p} [k_a (\mathbf{H}_a^2 - \mathbf{E}_a^2)] dV_p \cdot \int_V \mathbf{H} \cdot \mathbf{H}_a dV
\]

(A.197)

is obtained by applying

\[
\nabla \cdot (\mathbf{E}_a \times \mathbf{H}_a) = \mathbf{H}_a \cdot (\nabla \times \mathbf{E}_a) - \mathbf{E}_a \cdot (\nabla \times \mathbf{H}_a)
\]

\[
= k_a \mathbf{H}_a^2 - k_a \mathbf{E}_a^2.
\]

(A.198)
Figure A.1: Schematics of bead perturbation

Defining the initial resonant frequency as $\omega_0$ and final state frequency as $\omega$, the coefficients are represented as

$$\int_V H_0 \cdot H_a dV = C_1 e^{i\omega_0 t}$$  \hspace{1cm} (A.199)

$$\int_V H \cdot H_a dV = C_2 e^{i\omega t},$$  \hspace{1cm} (A.200)

where $C_1$ and $C_2$ are normalized terms independent of $t$. Applying (A.198), (A.199) and (A.200) to (A.195),

$$\epsilon \mu \frac{\partial^2}{\partial t^2} (C_2 e^{i\omega t}) + k_a^2 (C_2 e^{i\omega t}) = -k_a \int_{V_p} [k_a(H_a^2 - E_a^2)]dV_p(C_2 e^{i\omega t})$$

$$
\longleftrightarrow -\epsilon \mu \omega^2 + k_a^2 = -k_a^2 \int_{V_p} (H_a^2 - E_a^2) dV_p
$$  \hspace{1cm} (A.201)

is obtained. Applying $k_a^2 = \epsilon \mu \omega_0^2$ to (A.201),

$$\omega^2 = \omega_0^2 \left[ 1 + \int_{V_p} (H_a^2 - E_a^2)dV_p \right]$$  \hspace{1cm} (A.202)

is obtained. (A.202) is known as Slater’s Perturbation Formula.

Applying the definition (A.127), (A.128) and considering the normalization,

$$E_a^2 \rightarrow \frac{E_a^2}{\int_V E_a^2 dV} = \frac{eE_0^2}{\int_V eE_0^2 dV}$$  \hspace{1cm} (A.203)

$$H_a^2 \rightarrow \frac{H_a^2}{\int_V H_a^2 dV} = \frac{-\mu H_0^2}{\int_V \mu H_0^2 dV}$$  \hspace{1cm} (A.204)

is applied to calculate the actual field. Please note that normalization of both terms are carried out at phase zero, or at maximum. Therefore,

$$E_a^2 \rightarrow \frac{eE_0^2}{2U}$$  \hspace{1cm} (A.205)

$$H_a^2 \rightarrow \frac{\mu H_0^2}{2U}$$  \hspace{1cm} (A.206)
are applicable, where

\[ U = \frac{1}{2} \int_V \epsilon E^2 dV = \frac{1}{2} \int_V \mu H^2 dV \]  

(A.207)

is the stored energy of the cavity. Therefore,

\[ \omega^2 = \omega_0^2 \left[ 1 + \frac{\int_V (\mu H^2 - \epsilon E^2) dV}{2U} \right] \]  

(A.208)

is obtained.

A.5 Friss’s Formula

Noise Figure of an amplifier is defined as the ratio of S/N at input \((S_{in}/N_{in})\) and output \((S_{out}/N_{out})\), which is

\[ F \equiv \frac{S_{in}/N_{in}}{S_{out}/N_{out}} \]  

(A.209)

\[ NF \equiv 10 \log_{10} F. \]  

(A.210)

Representation NF is preferred, since it is measured by unit dB.

First, a single amplifier is considered. From (A.209), the output noise of the amplifier is calculated as

\[ N_{out} = \frac{S_{out}}{S_{in}} F N_{in} = G F N_{in}, \]  

(A.211)

where \( G = S_{out}/S_{in} \) is the gain of the amplifier. Assuming that the initial noise \( N_{in} \) is the thermal noise \( N_T \), this output noise is represented as

\[ N_{out} = G N_T + (F - 1) G N_T. \]  

(A.212)

If the amplifier was ideal, the output noise must be amplified with its gain \( G \), which is \( G N_T \). However due to finite \( F \), noise of the amplifier, \((F - 1) G N_T\) is added to ideal noise \( G N_T \).

Now, let’s consider the case of \( n \) pieces of amplifiers aligned in series. The output noise \( N_{out} \), noise figure \( F \) and gain \( G \) of \( n \)th amplifier are represented as \( N_n \), \( F_n \) and \( G_n \), respectively. Therefore, \( N_n \) is represented as

\[ N_n = G_n N_{n-1} + (F_n - 1) G_n N_T. \]  

(A.213)

Applying initial noise \( N_0 = N_T \), it is easily calculated as

\[ N_n = [G_1G_2 \cdots G_n F_1 + G_2G_3 \cdots G_n(F_2 - 1) + G_3 \cdots G_n(F_3 - 1) + \cdots + G_n(F_n - 1)] N_T. \]  

(A.214)

Applying (A.214) to (A.209),

\[ F_{total} = \frac{S_0/N_0}{S_n/N_n} \frac{N_n}{G_{total} N_T} = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1G_2} + \frac{F_4 - 1}{G_1G_2G_3} + \cdots + \frac{F_n - 1}{G_1G_2 \cdots G_{n-1}} \]  

(A.215)

is obtained. Of course, total gain \( G_{total} \) is \( G_1G_2 \cdots G_n \).
A.6 Standard Regression Coefficients

To evaluate contribution of regression parameters, standard regression coefficient is often used. Since parameters can be represented in arbitrary units, they cannot be simply compared. For example, parameter represented in length cannot be compared with the one represented in weight. In order to compare contribution among parameters of different units, all the parameters are normalized by Mean = 0 and RMS = 1. Thus standard regression coefficient is not affected by units.

In case of applying $N$ parameters, which $i$ th term is represented as $X_i$, regression analysis of a physical quantity $Y$ is represented as

$$Y = \Sigma_{i=0}^{N} b_i X_i,$$  \hspace{1cm} (A.216)

where $b_i$ is the coefficient of parameter $X_i$. In order to standardize the coefficients,

$$Y \rightarrow \frac{Y - \bar{Y}}{\sigma_Y},$$

$$X_i \rightarrow \frac{X_i - \bar{X_i}}{\sigma_{X_i}}$$  \hspace{1cm} (A.217)

are applied, where $\bar{A}$ represents the Mean of a physical quantity $A$, and $\sigma_A$ represents its standard deviation. Therefore, standard regression coefficients $b'_i$ are obtained as

$$\frac{Y - \bar{Y}}{\sigma_Y} = \Sigma_{i=0}^{N} b'_i \frac{X_i - \bar{X_i}}{\sigma_{X_i}},$$  \hspace{1cm} (A.218)

which leads to

$$b'_i = b_i \times \frac{\sigma_{X_i}}{\sigma_Y}.$$  \hspace{1cm} (A.219)

Constant term would be zero.

A.7 IP-BPM Design Drawing

The accurate design drawing of IP-BPM is shown in the next page.
Figure A.2: Design drawing of IP-BPM
References


[5] Y. Honda et al., Development of high resolution cavity beam position monitor, to be submitted.


[18] Drawn by Mr. Toshikazu Takatomi, Mechanical Engineering Center of KEK.